MAE 242

Three-Dimensional Kinematics

Roger E. Kaufman

Three-Dimensional Kinematics

Roger E. Kaufman

Notes based in part on lectures at the Summer Conference on Kinematics Theory and Practice, Massachusetts Institute of Technology, July 11-22, 1966

Allen S. Hall, Jr., George N. Sandor, Richard Hartenberg, Ferdinand Freudenstein, Jacques Denavit, Rudolf Beyer, Bernard Roth, C.H. Suh, An Tzu Yang, Joseph Beggs, and others too numerous too mention deserve most of the credit.

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Introduction to Spatial Kinematics

- *Any rigid body motion can be broken down into two parts:*

Introduction to Spatial Kinematics

- *Any rigid body motion can be broken down into two parts:*
  - *A rotation about some axis*

Introduction to Spatial Kinematics

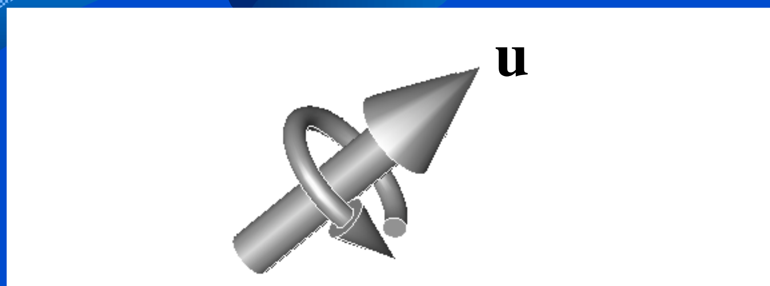
- Any rigid body motion can be broken down into two parts:
  - A rotation about some axis
  - A translation along the axis

We've been screwed!

Introduction to Spatial Kinematics

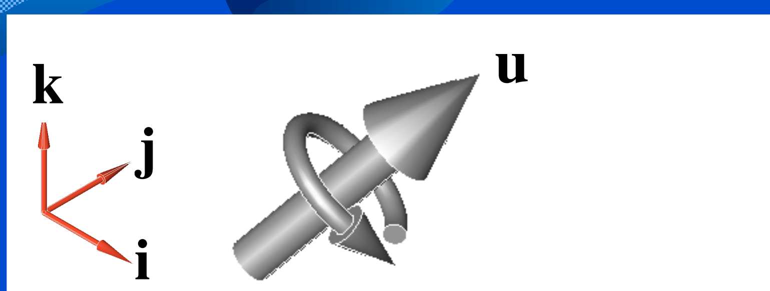
- First we will study the rotation and develop the "Finite Rotation Tensor"

Finite Rotations About a Known Axis of Rotation



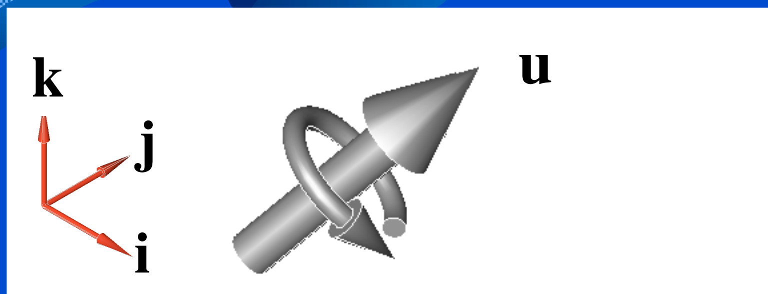
- Suppose the axis is specified by a unit vector ***u***

Finite Rotations About a Known Axis of Rotation



- Suppose the axis is specified by a unit vector ***u***
- Here, $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$

Finite Rotations About a Known Axis of Rotation



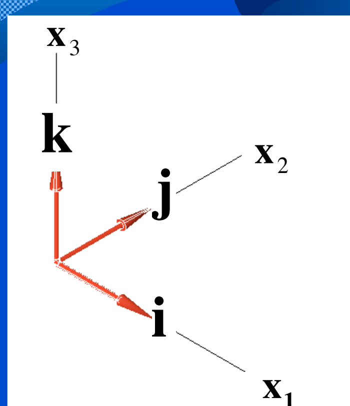
- *Here,*

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$$

$$|\mathbf{u}| = 1$$

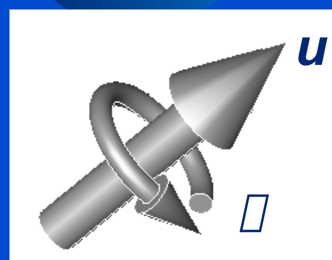
$$= \left(u_1^{\,2} + u_2^{\,2} + u_3^{\,2}\right)^{1/2}$$

Finite Rotations About a Known Axis of Rotation



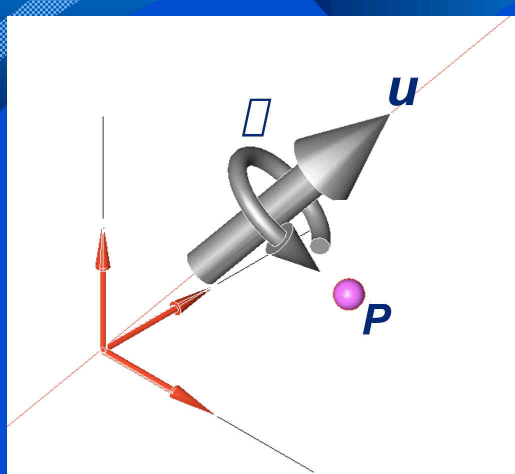
- *The convention is that the coordinate system is right-handed*

Finite Rotations About a Known Axis of Rotation



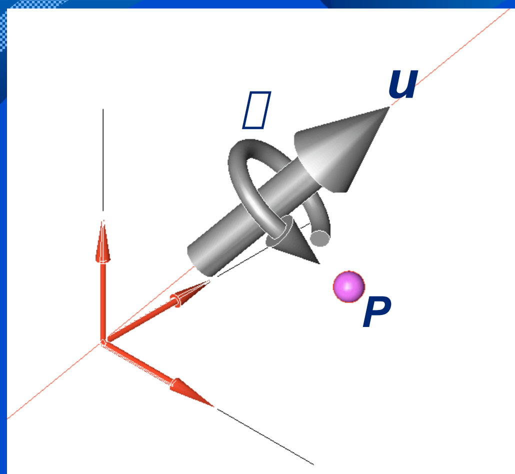
- The rotation angle about u is [illegible]

Finite Rotations About a Known Axis of Rotation



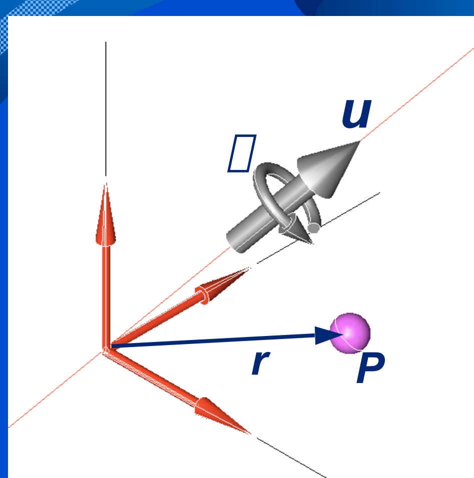
- Suppose the axis of rotation passes through the origin

Finite Rotations About a Known Axis of Rotation



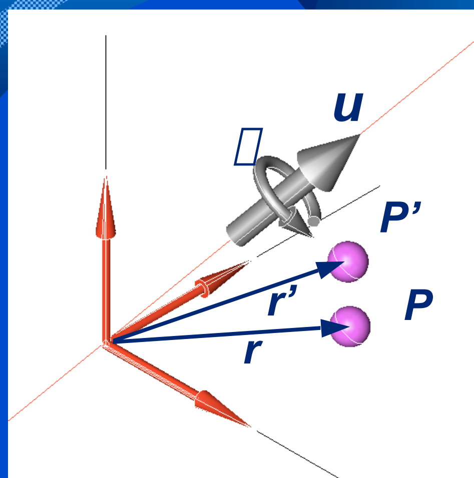
- Suppose we have a particle ***P***

Finite Rotations About a Known Axis of Rotation



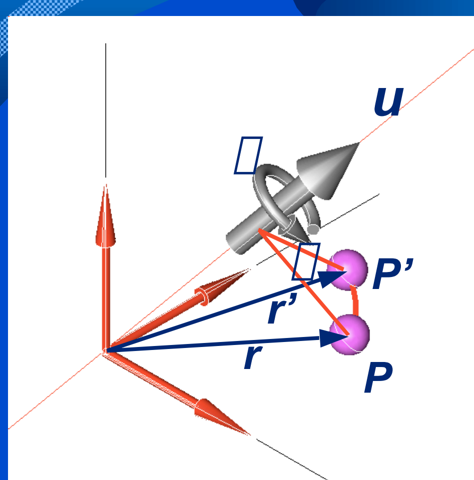
- Suppose we have a particle ***P***
- In its starting position it is located by the vector ***r***

Finite Rotations About a Known Axis of Rotation



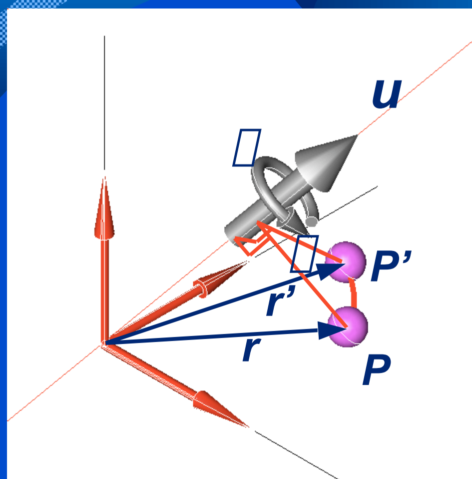
- In its ending position it is located by the vector **r’**

Finite Rotations About a Known Axis of Rotation



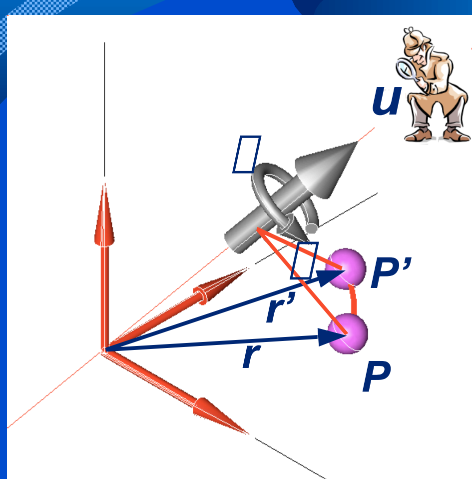
- In its ending position it is located by the vector **r’**
- The rotation angle was

Finite Rotations About a Known Axis of Rotation



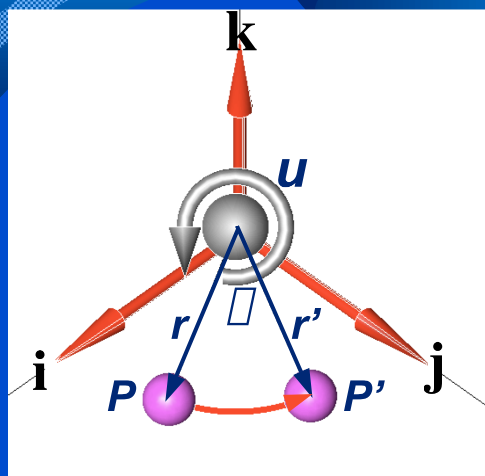
- The two radial lines shown are perpendicular to the rotation axis

Finite Rotations About a Known Axis of Rotation



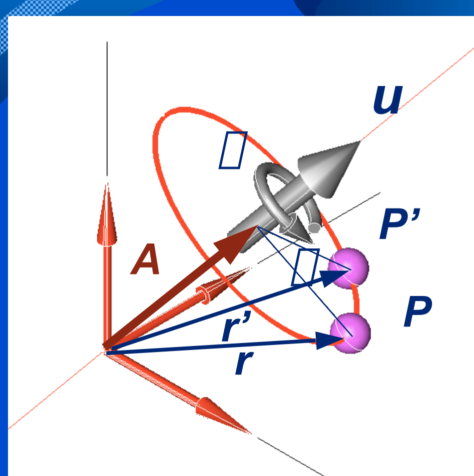
- Suppose we look in from the end to see the situation.

Finite Rotations About a Known Axis of Rotation



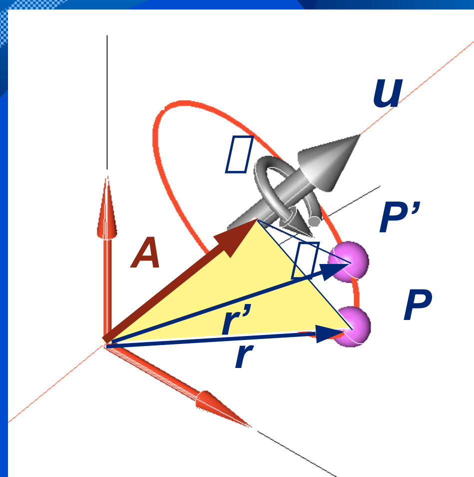
- *Looking in from the end of the axis vector, here's what we see.*
- *The particle has swung about **u** by the angle θ as it moved from **P** to **P'***

Finite Rotations About a Known Axis of Rotation



- *Let the vector from the origin to the plane of rotation of **P** be called **A***

Finite Rotations About a Known Axis of Rotation

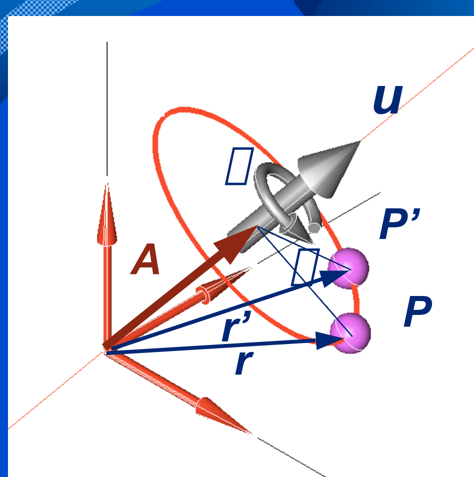


- *Vector **A** is then:*

$$\mathbf{A} = (\mathbf{u}\cdot\mathbf{r})\mathbf{u}$$

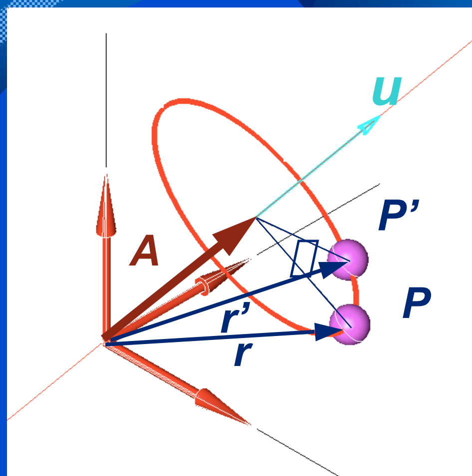
- *(Recall that **u** is a unit vector!)*

Finite Rotations About a Known Axis of Rotation



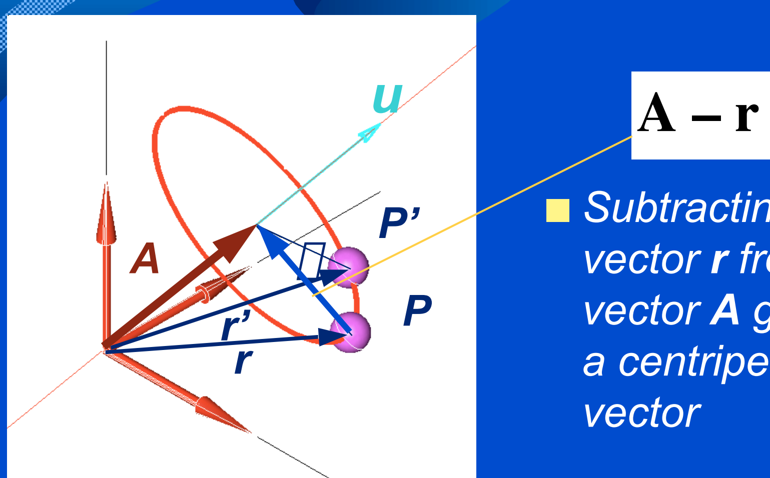
- *It will help if we establish a new rotated coordinate system at the tip of vector **A***

Finite Rotations About a Known Axis of Rotation



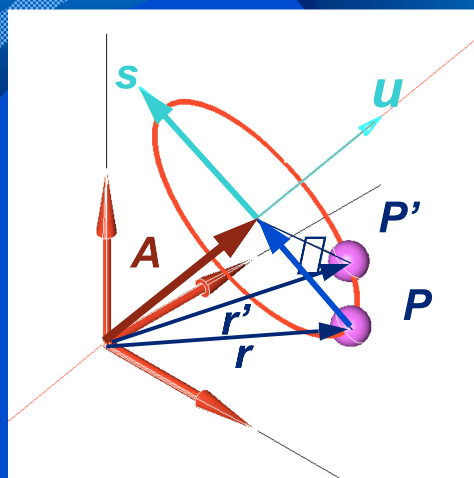
- Let’s slide ***u*** up to the tip of ***A***
- For clarity we’ll simplify ***u***’s representation.

Finite Rotations About a Known Axis of Rotation



- Subtracting vector ***r*** from vector ***A*** gives a centripetal vector

Finite Rotations About a Known Axis of Rotation

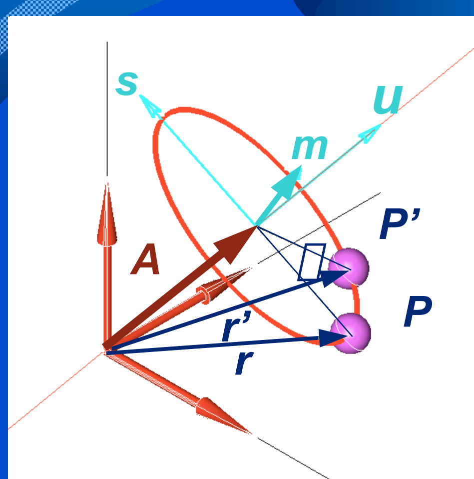


- We can make it into a free unit vector **s** by dividing that vector by its magnitude:

$$p = \left|\mathbf{r} - \mathbf{A}\right| \geq 0$$

$$\mathbf{s} = \frac{\mathbf{A} - \mathbf{r}}{p}$$

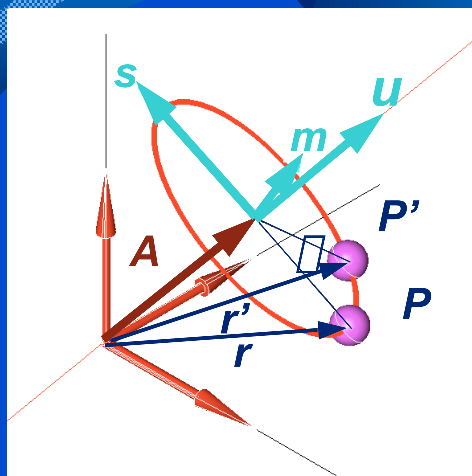
Finite Rotations About a Known Axis of Rotation



- Taking the cross product of **s** and **u** gives a third unit vector **m**

$$\mathbf{s} \times \mathbf{u} = \mathbf{m} = -\mathbf{u} \times \mathbf{s}$$

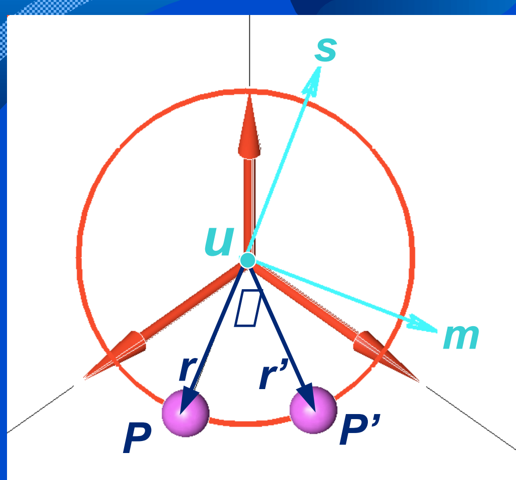
Finite Rotations About a Known Axis of Rotation



- Vectors **s, u, and m** form a right-hand triple.

$$\mathbf{s} \times \mathbf{u} = \mathbf{m} = -\mathbf{u} \times \mathbf{s}$$

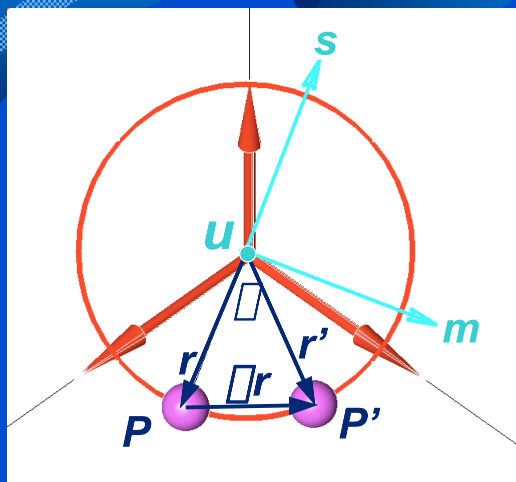
Finite Rotations About a Known Axis of Rotation



- Looking in down the ***u*** axis here's what we see

$$\mathbf{s} \times \mathbf{u} = \mathbf{m} = -\mathbf{u} \times \mathbf{s}$$

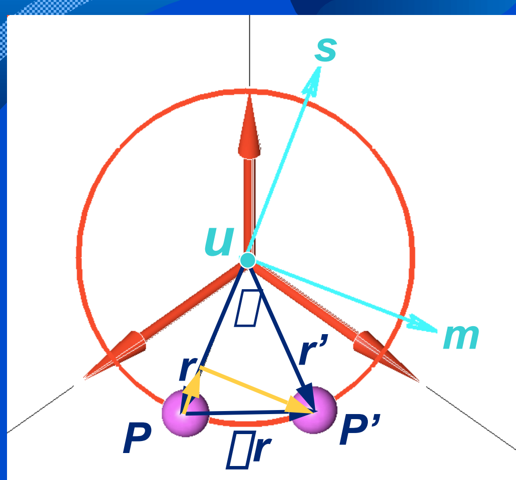
Finite Rotations About a Known Axis of Rotation



- Let

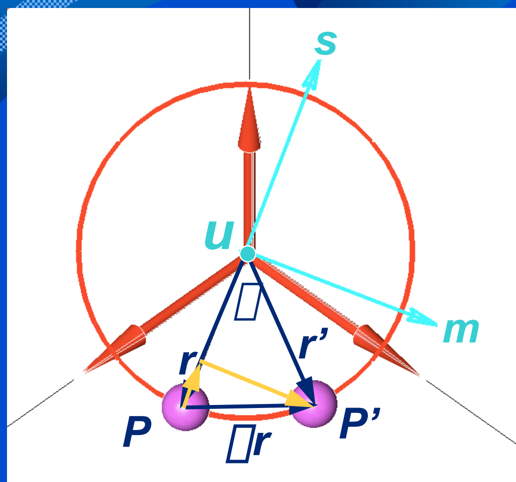
$$r' - r = \Delta r$$

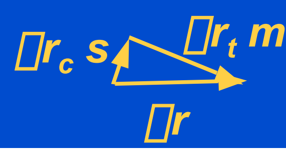
Finite Rotations About a Known Axis of Rotation



- Resolve Δ**r** into two components:
- One radially inward along **s**
- The other in the tangential direction along **m**

Finite Rotations About a Known Axis of Rotation





$$\Delta \mathbf{r} = \Delta r_c \mathbf{s} + \Delta r_t \mathbf{m}$$
$$= \underbrace{p(1 - \cos\theta)}_{\geq 0}\mathbf{s} + p\sin\theta\,\mathbf{m}$$

♬ Note: ♬

The radial component is always centripetal!

Finite Rotations About a Known Axis of Rotation

$$\Delta \mathbf{r} = \Delta r_c \mathbf{s} + \Delta r_t \mathbf{m}$$
$$= \underbrace{p(1 - \cos\theta)}_{\geq 0}\mathbf{s} + p\sin\theta\,\mathbf{m}$$

$$p\mathbf{m} = p\left(-\mathbf{u} \times \frac{\mathbf{A} - \mathbf{r}}{p}\right)$$

$$p\mathbf{m} = -\underbrace{\mathbf{u} \times \mathbf{A}}_{= 0 \text{ (since they have the same line of action)}} + \mathbf{u} \times \mathbf{r}$$

Finite Rotations About a Known Axis of Rotation

$$\Delta\mathbf{r} = \Delta r_c \mathbf{s} + \Delta r_t \mathbf{m}$$
$$= \underbrace{p(1 - \cos\theta)}_{\geq 0}\mathbf{s} + p\sin\theta\,\mathbf{m}$$

$$p\mathbf{m} = p\left(-\mathbf{u} \times \frac{\mathbf{A} - \mathbf{r}}{p}\right)$$
$$p\mathbf{m} = -\underbrace{\mathbf{u} \times \mathbf{A}}_{= 0 \text{ (since they have the same line of action)}} + \mathbf{u} \times \mathbf{r}$$

$$p\mathbf{s} = \mathbf{A} - \mathbf{r}$$

$$\Delta\mathbf{r} = (1 - \cos\theta)[\mathbf{A} - \mathbf{r}] + \sin\theta\,[\mathbf{u} \times \mathbf{r}]$$
$$= (1 - \cos\theta)[\mathbf{u}(\mathbf{u} \cdot \mathbf{r}) - \mathbf{r}] + \sin\theta\,[\mathbf{u} \times \mathbf{r}]$$
$$\mathbf{r}' = \mathbf{r} + \Delta\mathbf{r}$$

A brief digression…

Save this supersaturated thought while we have a brief digression to review some matrix algebra and stuff

Oy!

Representing Vectors

- Vectors can be represented by a column matrix:

$$\mathbf{A} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

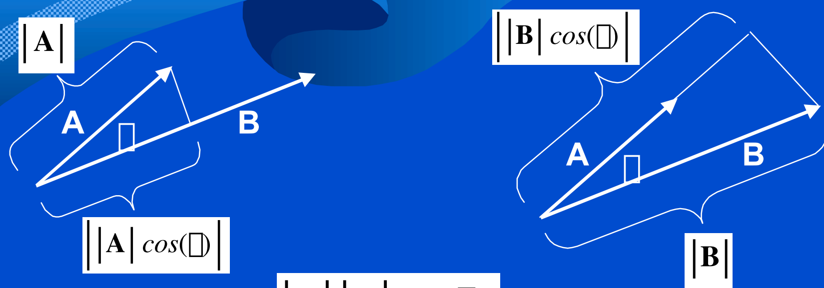
- Vectors can also be represented by a square matrix:

$$[A] = \begin{bmatrix} 0 & -A_3 & A_2 \\ A_3 & 0 & -A_1 \\ -A_2 & A_1 & 0 \end{bmatrix}$$

Refresher About Dot and Cross Products

- There are two kinds of vector products
  - Scalar or Dot Product
    - The result is a scalar
  - Cross Product (also called the Vector Product or the Outer Product)
    - The result is a vector

Dot Products

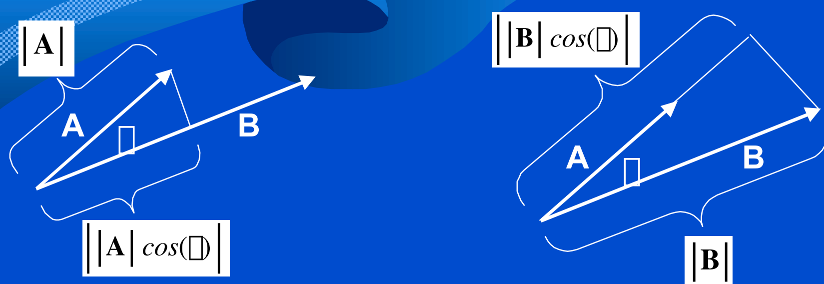


$|\mathbf{A}|\,|\mathbf{B}|\,cos(\square)$

$\mathbf{A} \bullet \mathbf{B} = \mathbf{B} \bullet \mathbf{A}$

- *The dot product of vectors* **A** *and* **B** *is written as* **A** · **B** *and is a scalar quantity.*
- *It is defined as the magnitude of* **A** *times the magnitude of* **B** *times the cosine of the angle between them.*

Dot Products



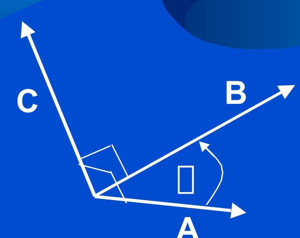
$|\mathbf{A}|\,|\mathbf{B}|\,cos(\square)$

- *Notice that the commutative law is true for the Dot Product:* $\mathbf{A} \bullet \mathbf{B} = \mathbf{B} \bullet \mathbf{A}$

Dot Product in Component Form

$$\mathbf{u} \bullet \mathbf{r} = \mathbf{u}_1\mathbf{r}_1 + \mathbf{u}_2\mathbf{r}_2 + \mathbf{u}_3\mathbf{r}_3$$

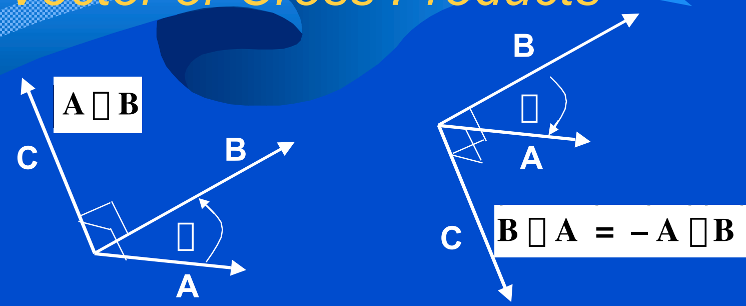
Vector or Cross Products



$$|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}| \sin(\theta)$$

- *The Cross Product of vectors* **A** *and* **B** *is a vector* **C** *which is perpendicular to the plane containing vectors* **A** *and* **B**
- *A right-handed screw turned from* **A** *towards* **B** *would advance in the direction of vector* **C**

Vector or Cross Products



- *Notice that the commutative law is NOT true for the Cross Product*

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

A Smattering of Cross Product Forms

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= [\mathrm{A}]\mathbf{B} \\ &= [\mathrm{A}]\begin{bmatrix} \mathrm{B}_1 \\ \mathrm{B}_2 \\ \mathrm{B}_3 \end{bmatrix} \\ &= [\mathrm{A}]\,[\mathrm{B}] - [\mathrm{B}]\,[\mathrm{A}]\end{aligned}$$

Cross Product in Determinant Form

$$\mathbf{u} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ r_1 & r_2 & r_3 \end{vmatrix}$$

$$= (u_2 r_3 - u_3 r_2)\,\mathbf{i} + (u_3 r_1 - u_1 r_3)\,\mathbf{j} + (u_1 r_2 - u_2 r_1)\,\mathbf{k}$$

A brief digression…

Now, where were we???

You were giving me indigression!

Finite Rotations About a Known Axis of Rotation

$$\Delta\mathbf{r} = (1 - \cos\theta)[\mathbf{A} - \mathbf{r}] + \sin\theta\,[\mathbf{u} \times \mathbf{r}]$$
$$= (1 - \cos\theta)[\mathbf{u}(\mathbf{u}\cdot\mathbf{r}) - \mathbf{r}] + \sin\theta\,[\mathbf{u} \times \mathbf{r}]$$
$$\mathbf{r}' = \mathbf{r} + \Delta\mathbf{r}$$

- *Since (1 - cos θ) crops up repeatedly it will save lots of work to call this "versine (θ)"*
- *I would copyright the name except that vers (θ) is how mathematicians have always referred to (1 - cos θ)*

Finite Rotations About a Known Axis of Rotation

$$\mathbf{r}' = \mathbf{r} + \Delta\mathbf{r}$$

$$\begin{bmatrix} \mathbf{r}'_1 \\ \mathbf{r}'_2 \\ \mathbf{r}'_3 \end{bmatrix} = \left\{ \begin{bmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{bmatrix} + \begin{bmatrix} 0 & -\sin\theta\,\mathbf{u}_3 & +\sin\theta\,\mathbf{u}_2 \\ +\sin\theta\,\mathbf{u}_3 & 0 & -\sin\theta\,\mathbf{u}_1 \\ -\sin\theta\,\mathbf{u}_2 & +\sin\theta\,\mathbf{u}_1 & 0 \end{bmatrix} + \begin{bmatrix} vers\,\theta\,\mathbf{u}_1^2 & vers\,\theta\,\mathbf{u}_1\mathbf{u}_2 & vers\,\theta\,\mathbf{u}_1\mathbf{u}_3 \\ vers\,\theta\,\mathbf{u}_2\mathbf{u}_1 & vers\,\theta\,\mathbf{u}_2^2 & vers\,\theta\,\mathbf{u}_2\mathbf{u}_3 \\ vers\,\theta\,\mathbf{u}_3\mathbf{u}_1 & vers\,\theta\,\mathbf{u}_3\mathbf{u}_2 & vers\,\theta\,\mathbf{u}_3^2 \end{bmatrix} \right\} \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}$$

Finite Rotations About a Known Axis of Rotation

or

$$\mathbf{r}' = [\mathbf{R}]\mathbf{r}$$

where

$$[\mathbf{R}] =$$

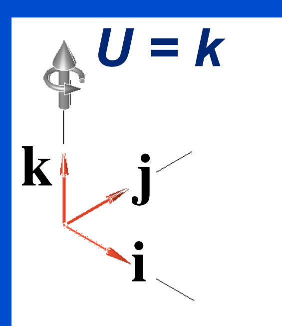
$$\begin{bmatrix} vers\,\theta\, u_1^2 + \cos\theta & vers\,\theta\, u_1 u_2 - \sin\theta\, u_3 & vers\,\theta\, u_1 u_3 + \sin\theta\, u_2 \\ vers\,\theta\, u_2 u_1 + \sin\theta\, u_3 & vers\,\theta\, u_2^2 + \cos\theta & vers\,\theta\, u_2 u_3 - \sin\theta\, u_1 \\ vers\,\theta\, u_3 u_1 - \sin\theta\, u_2 & vers\,\theta\, u_3 u_2 + \sin\theta\, u_1 & vers\,\theta\, u_3^2 + \cos\theta \end{bmatrix}$$

- *We can call this matrix the "Finite Rotation Tensor*
- *It is a matrix operator that converts a vector* **r** *into a vector* **r'** *when the origin is on the axis of rotation*

Example: Rotation About the Z Axis

- Suppose the axis of rotation is the + Z axis

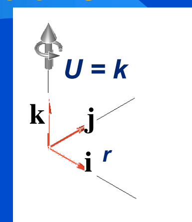
$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



$$\mathbf{u}_3 = 1, \quad \mathbf{u}_1 = \mathbf{u}_2 = 0$$

Example: Rotation About the Z Axis

- From the general expression:



$u_3 = 1, \; u_1 = u_2 = 0$

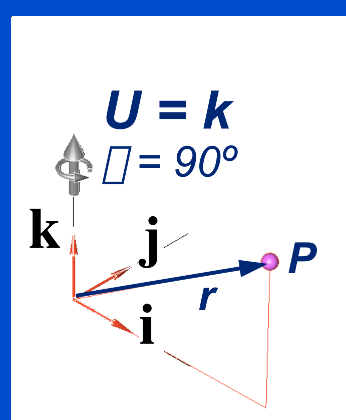
$$[R] = \begin{bmatrix} vers\,\theta\, u_1^2 + cos\,\theta & vers\,\theta\, u_1 u_2 - sin\,\theta\, u_3 & vers\,\theta\, u_1 u_3 + sin\,\theta\, u_2 \\ vers\,\theta\, u_2 u_1 + sin\,\theta\, u_3 & vers\,\theta\, u_2^2 + cos\,\theta & vers\,\theta\, u_2 u_3 - sin\,\theta\, u_1 \\ vers\,\theta\, u_3 u_1 - sin\,\theta\, u_2 & vers\,\theta\, u_3 u_2 + sin\,\theta\, u_1 & vers\,\theta\, u_3^2 + cos\,\theta \end{bmatrix}$$

$$[R_Z] = \begin{bmatrix} cos\,\theta & -sin\,\theta & 0 \\ sin\,\theta & cos\,\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example: Rotation By 90º About the Z Axis

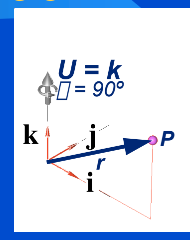
- Suppose the angle of rotation is 90º
- Also, let the starting point be located at

$$r = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$



Example: Rotation By 90º About the Z Axis

- From the general expression:



$\mathbf{u}_3 = 1, \ \mathbf{u}_1 = \mathbf{u}_2 = 0$

$$[\mathbf{R}] = \begin{bmatrix} vers\,\theta\, u_1^2 + cos\,\theta & vers\,\theta\, u_1 u_2 - sin\,\theta\, u_3 & vers\,\theta\, u_1 u_3 + sin\,\theta\, u_2 \\ vers\,\theta\, u_2 u_1 + sin\,\theta\, u_3 & vers\,\theta\, u_2^2 + cos\,\theta & vers\,\theta\, u_2 u_3 - sin\,\theta\, u_1 \\ vers\,\theta\, u_3 u_1 - sin\,\theta\, u_2 & vers\,\theta\, u_3 u_2 + sin\,\theta\, u_1 & vers\,\theta\, u_3^2 + cos\,\theta \end{bmatrix}$$

$$[\mathbf{R}_Z] = \begin{bmatrix} cos\,\theta & -sin\,\theta & 0 \\ sin\,\theta & cos\,\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example: Rotation By 90º About the Z Axis

$$cos\,90 = 0$$

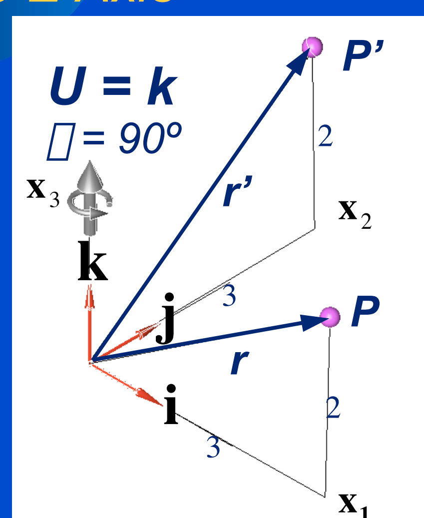
$$sin\,90 = 1$$

$$[\mathbf{R}_Z] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{r}' = [\mathbf{R}]\mathbf{r}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{3} \\ \mathbf{0} \\ \mathbf{2} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} \\ \mathbf{3} \\ \mathbf{2} \end{bmatrix}$$



Composition of Rotations

- *Suppose you have a rotation $[R_{12}]$ which is followed by a rotation $[R_{23}]$*
- *How can we find a single rotation $[R_{13}]$ which is equivalent to this sequence?*

Composition of Rotations

- *Given: $[R_{12}]$ followed by $[R_{23}]$*
- *Find: $[R_{13}]$*
- *Procedure:*

$$\mathbf{r}_2 = [R_{12}]\,\mathbf{r}_1$$

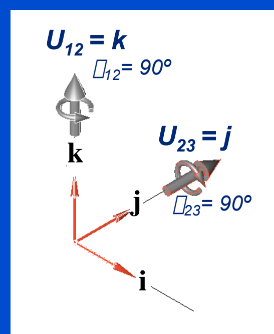
$$\mathbf{r}_3 = [R_{23}]\,\mathbf{r}_2 = [R_{23}]\,[R_{12}]\,\mathbf{r}_1$$

♫ Note: ♫

The order of the factors is important!
Spatial rotations don't commute!

Composition of Rotations Example

- *Suppose we have a 90º rotation about the Z axis followed by a 90º rotation about the Y axis:*



Composition of Rotations Example

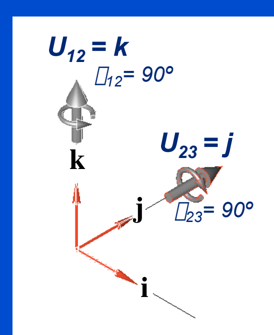
$$[\mathbf{R}_{12}] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\mathbf{R}_{23}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$[\mathbf{R}_{13}] = [\mathbf{R}_{23}]\,[\mathbf{R}_{12}]$$

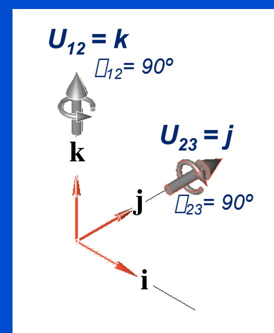
$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\mathbf{R}_{13}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

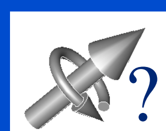


Equivalent Single Rotation

$$[\mathbf{R}_{13}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

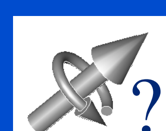


- *How can we find the equivalent single rotation this tensor represents?*



Equivalent Single Rotation

- *We have nine equations available to us*
- *We can equate each element of this tensor to the corresponding term in the general finite rotation tensor*

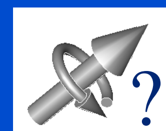


$$[\mathbf{R}_{13}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$[\mathbf{R}] = \begin{bmatrix} vers\,\phi\,u_1^2 + \cos\phi & vers\,\phi\,u_1u_2 - \sin\phi\,u_3 & vers\,\phi\,u_1u_3 + \sin\phi\,u_2 \\ vers\,\phi\,u_2u_1 + \sin\phi\,u_3 & vers\,\phi\,u_2^2 + \cos\phi & vers\,\phi\,u_2u_3 - \sin\phi\,u_1 \\ vers\,\phi\,u_3u_1 - \sin\phi\,u_2 & vers\,\phi\,u_3u_2 + \sin\phi\,u_1 & vers\,\phi\,u_3^2 + \cos\phi \end{bmatrix}$$

Equivalent Single Rotation

- We also know that the magnitude of the **u** vector is 1, so we have more equations than are needed

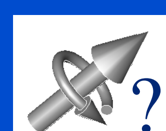


$$[\mathbf{R}_{13}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$[\mathbf{R}] = \begin{bmatrix} vers\,\theta\, u_1^2 + \cos\theta & vers\,\theta\, u_1 u_2 - \sin\theta\, u_3 & vers\,\theta\, u_1 u_3 + \sin\theta\, u_2 \\ vers\,\theta\, u_2 u_1 + \sin\theta\, u_3 & vers\,\theta\, u_2^2 + \cos\theta & vers\,\theta\, u_2 u_3 - \sin\theta\, u_1 \\ vers\,\theta\, u_3 u_1 - \sin\theta\, u_2 & vers\,\theta\, u_3 u_2 + \sin\theta\, u_1 & vers\,\theta\, u_3^2 + \cos\theta \end{bmatrix}$$

Equivalent Single Rotation

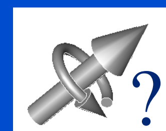
- Look, for example, at the diagonal elements:



$$[\mathbf{R}_{13}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$[\mathbf{R}] = \begin{bmatrix} vers\,\theta\, u_1^2 + \cos\theta & vers\,\theta\, u_1 u_2 - \sin\theta\, u_3 & vers\,\theta\, u_1 u_3 + \sin\theta\, u_2 \\ vers\,\theta\, u_2 u_1 + \sin\theta\, u_3 & vers\,\theta\, u_2^2 + \cos\theta & vers\,\theta\, u_2 u_3 - \sin\theta\, u_1 \\ vers\,\theta\, u_3 u_1 - \sin\theta\, u_2 & vers\,\theta\, u_3 u_2 + \sin\theta\, u_1 & vers\,\theta\, u_3^2 + \cos\theta \end{bmatrix}$$

Equivalent Single Rotation



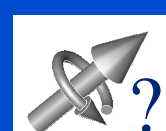
- *Look, for example, at the diagonal elements:*

$$[\mathbf{R}_{13}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$[\mathbf{R}] = \begin{bmatrix} vers\,\theta\, u_1^2 + \cos\theta & vers\,\theta\, u_1 u_2 - \sin\theta\, u_3 & vers\,\theta\, u_1 u_3 + \sin\theta\, u_2 \\ vers\,\theta\, u_2 u_1 + \sin\theta\, u_3 & vers\,\theta\, u_2^2 + \cos\theta & vers\,\theta\, u_2 u_3 - \sin\theta\, u_1 \\ vers\,\theta\, u_3 u_1 - \sin\theta\, u_2 & vers\,\theta\, u_3 u_2 + \sin\theta\, u_1 & vers\,\theta\, u_3^2 + \cos\theta \end{bmatrix}$$

$$vers\,\theta\, u_m^2 + \cos\theta = 0\,, \quad m = 1, 2, 3$$

Equivalent Single Rotation



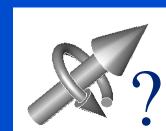
$$vers\,\theta\, u_m^2 + \cos\theta = 0\,, \quad m = 1, 2, 3$$

or

$$u_m^2 = \frac{-\cos\theta}{vers\,\theta}\,, \quad m = 1, 2, 3$$

Equivalent Single Rotation

- Also, remember that **u** is a unit vector:



$$\sum_{m=1}^{3} u_{13m}^2 = 1$$

$$u_{13}^2 = \frac{1}{3}$$

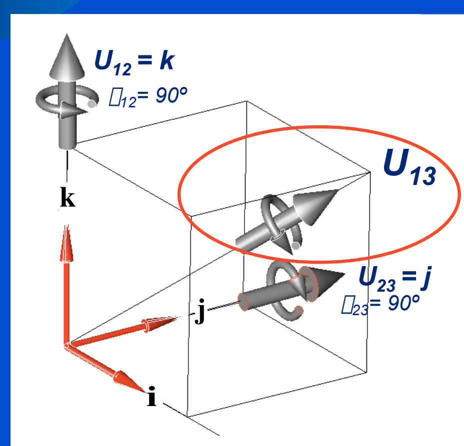
$$u_{13m} = \pm\sqrt{\frac{1}{3}}$$

$$\mathbf{u}_{13} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \text{ or } \begin{bmatrix} \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{bmatrix}$$

Equivalent Single Rotation

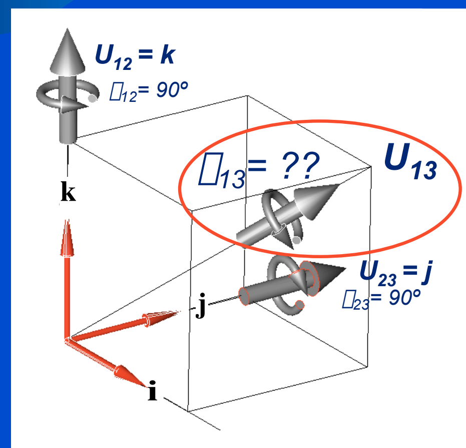
- So the axis **u** lies on the diagonal:

$$\mathbf{u}_{13} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \text{ or } \begin{bmatrix} \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{bmatrix}$$



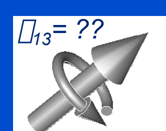
Equivalent Single Rotation

- *What is the equivalent rotation angle θ_{13}?*



Equivalent Single Rotation

- *To find angle θ_{13} look at off-diagonal elements such as (3, 2) and (2, 3):*

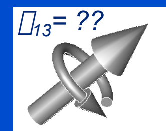


$$[\mathbf{R}_{13}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$[\mathbf{R}] = \begin{bmatrix} vers\,\theta\, u_1^2 + cos\,\theta & vers\,\theta\, u_1 u_2 - sin\,\theta\, u_3 & vers\,\theta\, u_1 u_3 + sin\,\theta\, u_2 \\ vers\,\theta\, u_2 u_1 + sin\,\theta\, u_3 & vers\,\theta\, u_2^2 + cos\,\theta & vers\,\theta\, u_2 u_3 - sin\,\theta\, u_1 \\ vers\,\theta\, u_3 u_1 - sin\,\theta\, u_2 & vers\,\theta\, u_3 u_2 + sin\,\theta\, u_1 & vers\,\theta\, u_3^2 + cos\,\theta \end{bmatrix}$$

Equivalent Single Rotation

- To find angle θ_{13} look at off-diagonal elements such as (3, 2) and (2, 3):



(3, 2) $\quad vers\,\theta\, u_2 u_3 - sin\,\theta\, u_1 = 1$

(2, 3) $\quad vers\,\theta\, u_3 u_2 + sin\,\theta\, u_1 = 0$

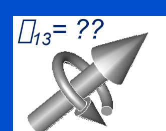
(3, 2) - (2, 3):

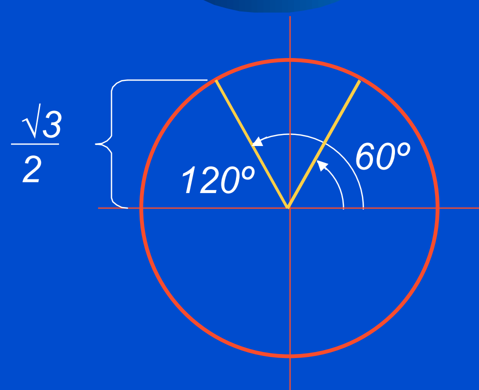
$$2\, sin\,\theta\, u_1 = 1$$

$$sin\,\theta = \frac{1}{2\,u_1} = \frac{\sqrt{3}}{2}$$

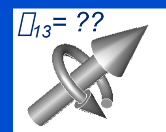
Equivalent Single Rotation Angle

- Therefor, θ_{13} is either 60° or 120°





Equivalent Single Rotation Angle



- Is θ_{13} 60º or 120º?

(3, 2) + (2, 3):

$$2\,(1 - \cos\theta)\,u_2 u_3 = 1$$

$$\frac{2}{3}(1 - \cos\theta) = 1$$

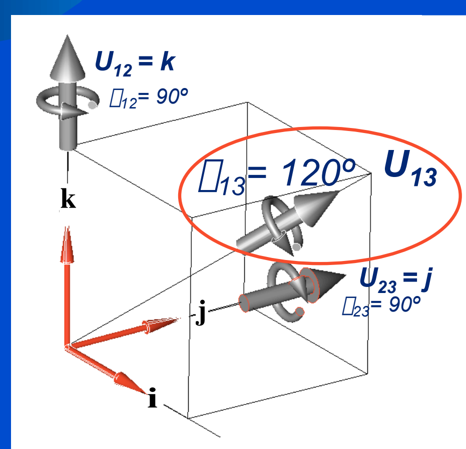
$$1 - \cos\theta = \frac{3}{2}$$

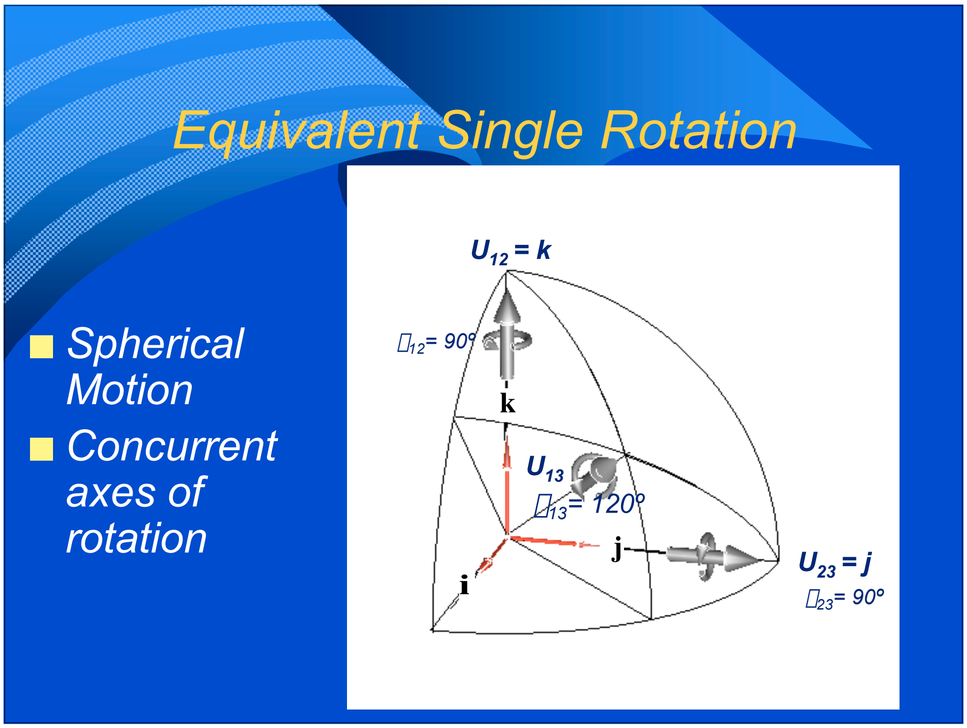
$$\cos\theta = -\frac{1}{2}$$

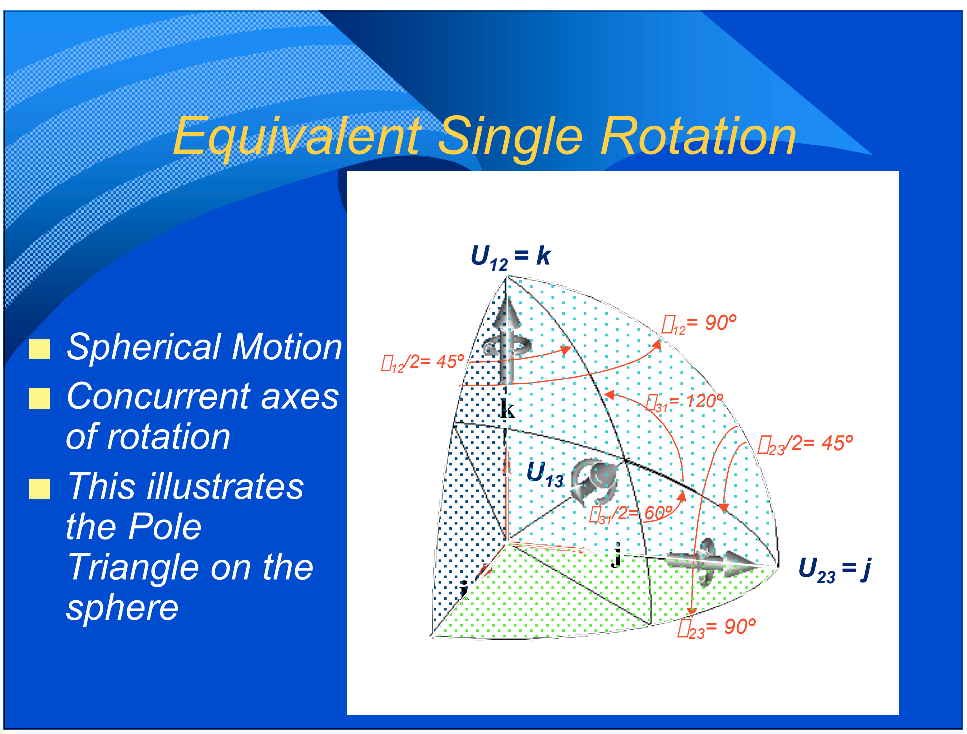
$$\therefore \theta = 120^{o}$$

Equivalent Single Rotation

- This is a Spherical Motion!
- All the axes of rotation are concurrent




Equivalent Single Rotation
Spherical Motion
Concurrent axes of rotation
$U_{12} = k$
□12= 90°
k
U_{13}
□13= 120°
j
i
$U_{23} = j$
□23= 90°


Equivalent Single Rotation
Spherical Motion
Concurrent axes of rotation
This illustrates the Pole Triangle on the sphere
$U_{12} = k$
□12/2= 45°
□12= 90°
□31= 120°
□23/2= 45°
k
U_{13}
□31/2= 60°
j
i
$U_{23} = j$
□23= 90°

Properties of the Rotation Matrix

- Meaning of the columns of ***[R]:***

$$[\mathbf{R}] = \begin{bmatrix} vers\,\theta\, u_1^2 + \cos\theta & vers\,\theta\, u_1 u_2 - \sin\theta\, u_3 & vers\,\theta\, u_1 u_3 + \sin\theta\, u_2 \\ vers\,\theta\, u_2 u_1 + \sin\theta\, u_3 & vers\,\theta\, u_2^2 + \cos\theta & vers\,\theta\, u_2 u_3 - \sin\theta\, u_1 \\ vers\,\theta\, u_3 u_1 - \sin\theta\, u_2 & vers\,\theta\, u_3 u_2 + \sin\theta\, u_1 & vers\,\theta\, u_3^2 + \cos\theta \end{bmatrix}$$

Properties of the Rotation Matrix

- What is the meaning of the columns of ***[R]?***

$$\mathbf{r}' = [\mathbf{R}]\mathbf{r}$$

- Multiply ***[R]*** by the identity matrix

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\mathbf{R}][\mathbf{I}] = [\mathbf{R}]$$

Properties of the Rotation Matrix

$$\text{Let } [\mathbf{R}] = [\mathbf{a}_{ij}] = \begin{bmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} & \mathbf{a}_{13} \\ \mathbf{a}_{21} & \mathbf{a}_{22} & \mathbf{a}_{23} \\ \mathbf{a}_{31} & \mathbf{a}_{32} & \mathbf{a}_{33} \end{bmatrix}$$

- *Now, let **[R]** operate on the **i unit vector***

$$[\mathbf{R}]\,\mathbf{i} = [\mathbf{R}]\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}a_{11}\\a_{21}\\a_{31}\end{bmatrix}$$

Properties of the Rotation Matrix

$$[\mathbf{R}]\,\mathbf{i} = [\mathbf{R}]\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}a_{11}\\a_{21}\\a_{31}\end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \mathbf{r}' = \begin{bmatrix}a_{11}\\a_{21}\\a_{31}\end{bmatrix} = \mathbf{v}_1$$

*The **i unit vector before and after rotation***

Properties of the Rotation Matrix

$$\mathbf{r} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{r}' = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \mathbf{v}_1$$

Since this is a pure rotation about a concurrent axis

$$|\mathbf{r}| = |\mathbf{r}'|$$

$$\mathbf{a}_{11}^2 + \mathbf{a}_{21}^2 + \mathbf{a}_{31}^2 = |\mathbf{v}_1^2| = 1$$

Properties of the Rotation Matrix

$$[\mathbf{R}]\,\mathbf{i} = [\mathbf{R}] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{r}' = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \mathbf{v}_1$$

So $\boldsymbol{V_1}$ (the first column of the rotation matrix) is also a unit vector

Properties of the Rotation Matrix

$$[\mathbf{R}]\,\mathbf{i} = [\mathbf{R}]\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}a_{11}\\a_{21}\\a_{31}\end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \mathbf{r}' = \begin{bmatrix}a_{11}\\a_{21}\\a_{31}\end{bmatrix} = \mathbf{v}_1$$

*Further, the first column of the rotation matrix is simply the rotated **i** unit vector*

Properties of the Rotation Matrix

- *Similarly, the second and third columns of the rotation matrix correspond to the rotated **j** and **k** unit vectors*
- *So*

$$[\mathbf{R}] = [\mathbf{v}_1\,,\ \mathbf{v}_2\,,\ \mathbf{v}_3]$$

Properties of the Rotation Matrix

- *You can partition the rotation matrix into three unit vectors corresponding to the original base unit vectors:*

$$[\mathbf{R}] = \left[\begin{array}{c:c:c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right]$$

Properties of the Rotation Matrix

$$[\mathbf{R}] = [\mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3]$$

- *Also, since the column vectors* $\boldsymbol{V_1}$ *,* $\boldsymbol{V_2}$ *, and* $\boldsymbol{V_3}$ *correspond to the original set of base unit vectors after rotation they form an orthonormal right-handed triple*

Properties of the Rotation Matrix

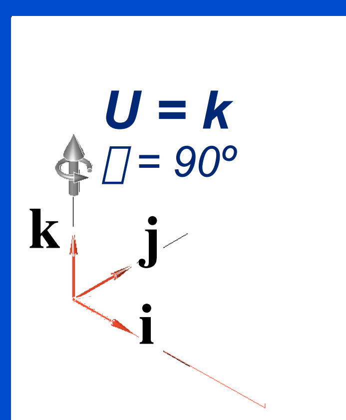
$$[R] = [v_1 , v_2 , v_3]$$

- *Also, since the column vectors* $\boldsymbol{V_1}$ *,* $\boldsymbol{V_2}$ *, and* $\boldsymbol{V_3}$ *correspond to the original set of base unit vectors after rotation they form an orthonormal right-handed triple*

Example: Check for Rotation By 90º About the Z Axis

- *By inspection:*

$$[R] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Properties of the Rotation Matrix

$$[R] = [v_1 \, , v_2 \, , v_3]$$

- V_1 , V_2 , and V_3 form an orthonormal right-handed triple:

$$V_1 \times V_2 = V_3$$
$$V_2 \times V_3 = V_1$$
$$V_3 \times V_1 = V_2$$

Properties of the Rotation Matrix

- The vector triple products of V_1 , V_2 , and V_3 have the properties:

$$V_1 \bullet (V_2 \times V_3) = 1$$
$$V_2 \bullet (V_3 \times V_1) = 1$$
$$V_3 \bullet (V_1 \times V_2) = 1$$

Properties of the Rotation Matrix

- *The dot and the cross can be interchanged in these vector triple product equations (moving the parentheses, of course)*

$$\mathbf{V_1} \bullet (\mathbf{V_2} \times \mathbf{V_3}) = \mathbf{1}$$
$$\mathbf{V_2} \bullet (\mathbf{V_3} \times \mathbf{V_1}) = \mathbf{1}$$
$$\mathbf{V_3} \bullet (\mathbf{V_1} \times \mathbf{V_2}) = \mathbf{1}$$

Properties of the Rotation Matrix

- *Also, the determinant of* ***[R]*** *is unity:*

$$\mathrm{Det}\,[\mathbf{R}] = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \mathbf{1}$$

Properties of the Rotation Matrix

- *Proof:*

$$\mathbf{V}_2 \times \mathbf{V}_3 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} + \mathbf{j}\begin{vmatrix} a_{32} & a_{12} \\ a_{33} & a_{13} \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \end{vmatrix}$$

Properties of the Rotation Matrix

$$\mathbf{V}_1 \bullet \mathbf{V}_2 \times \mathbf{V}_3 = 1$$

$$= \mathbf{a}_{11}\,(x \text{ component of } \mathbf{V}_2 \times \mathbf{V}_3)$$

$$+ \mathbf{a}_{21}\,(y \text{ component of } \mathbf{V}_2 \times \mathbf{V}_3)$$

$$+ \mathbf{a}_{31}\,(z \text{ component of } \mathbf{V}_2 \times \mathbf{V}_3)$$

$$= \text{Det}\,[\mathbf{R}] = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= 1$$

Q.E.D.

Inverse of the Rotation Matrix:

- *Here's the algorithm for finding the inverse of a general matrix:*
  - *Replace each element of the matrix by its cofactor (the signed minor of the element)*
  - *Transpose rows and columns (this yields the adjoint of the matrix)*
  - *Divide by the determinant of the matrix (this yields the inverse of the matrix)*

Inverse of the Rotation Matrix:

- *Note: For a finite rotation matrix*
  - *Each element is equal to its own cofactor*
  - *The determinant of a rotation matrix is equal to 1*

Inverse of the Rotation Matrix:

- *For a finite rotation matrix*
  - *Each element is equal to its cofactor*
  - *The determinant of the rotation matrix is equal to 1*
- *Thus:*

$$[\mathbf{R}]^{-1} = \frac{[\mathbf{R}]^T}{|\mathbf{R}|} = [\mathbf{R}]^T$$

$$[\mathbf{R}][\mathbf{R}]^{-1} = [\mathbf{I}]$$

Example: Finding the Inverse of a Rotation Matrix:

$$[\mathbf{R}] = [\mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3]$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Example: Finding the Inverse of a Rotation Matrix:

$$\text{Cofactor}\left[\mathbf{R}\right] = \begin{bmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{31} & a_{33} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{12} & a_{13} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{31} & a_{32} \\ a_{11} & a_{12} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{23} \\ a_{11} & a_{13} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}$$

(Replacing each element by its cofactor produces no change since this is a direction cosine matrix)

Example: Finding the Inverse of a Rotation Matrix:

$$\text{Cofactor}\left[\mathbf{R}\right]^{T} = \begin{bmatrix} \begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{12} \\ a_{33} & a_{13} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{21} \\ a_{33} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{31} \\ a_{13} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{11} \\ a_{23} & a_{13} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{31} & a_{11} \\ a_{32} & a_{12} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} \end{bmatrix}$$

Example: Finding the Inverse of a Rotation Matrix:

$$\text{Cofactor}[\mathbf{R}]^T = \begin{bmatrix} \begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{12} \\ a_{33} & a_{13} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{21} \\ a_{33} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{31} \\ a_{13} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{11} \\ a_{23} & a_{13} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{31} & a_{11} \\ a_{32} & a_{12} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} \end{bmatrix}$$

This is the adjoint of the rotation matrix.

(The transpose of a general cofactor matrix is called the adjoint of the matrix.)

Example: Finding the Inverse of a Rotation Matrix:

$$\text{Cofactor}[\mathbf{R}]^T = \begin{bmatrix} \begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{12} \\ a_{33} & a_{13} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{21} \\ a_{33} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{31} \\ a_{13} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{11} \\ a_{23} & a_{13} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{31} & a_{11} \\ a_{32} & a_{12} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} \end{bmatrix}$$

This is the adjoint of the rotation matrix.

Dividing this by det[R] (which happens to be 1 in this case!) gives the inverse. Let's see if this is all true.

Example: Finding the Inverse of a Rotation Matrix:

Compare the first row of Cofactor$[\mathbf{R}]^T$ with the components of $\mathbf{V}_2 \times \mathbf{V}_3$:

$$\mathbf{V}_2 \times \mathbf{V}_3 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = \mathbf{V}_1$$

Example: Finding the Inverse of a Rotation Matrix:

$$\text{so Cofactor}[\mathbf{R}]^T = \begin{bmatrix} V_{1x} & V_{1Y} & V_{1Z} \\ V_{2x} & V_{2Y} & V_{2Z} \\ V_{3x} & V_{3Y} & V_{3Z} \end{bmatrix}$$

Example: Finding the Inverse of a Rotation Matrix:

$$\square \; [\mathbf{R}]^{-1} = \begin{bmatrix} \mathbf{V}_1^{\mathbf{T}} \\ \mathbf{V}_2^{\mathbf{T}} \\ \mathbf{V}_3^{\mathbf{T}} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Resolution of Rotations:

- *Suppose we are given a rotation* ***[R]***
- *Can we resolve it into a series of two successive rotations?*
- *How?*

Resolution of Rotations:

Given a rotation $[\mathbf{R}]$ (with a given $\mathbf{u}$ and □)

find rotations $[\mathbf{R}_1]$ (with a $\mathbf{u}_1$ and $□_1$)

and $[\mathbf{R}_2]$ (with a $\mathbf{u}_2$ and $□_2$)

so that $[\mathbf{R}] = [\mathbf{R}_2][\mathbf{R}_1]$

Resolution of Rotations:

$$[\mathbf{R}] = [\mathbf{R}_2][\mathbf{R}_1]$$

- *What are the unknowns?*
  - $\mathbf{u}_1$ and $□_1$)
  - $\mathbf{u}_2$ and $□_2$)
- *We have eight scalar unknowns:*

$$u_{1_x}, u_{1_y}, u_{1_z}, □_1$$
$$u_{2_x}, u_{2_y}, u_{2_z}, □_2$$

Resolution of Rotations:

$$[\mathbf{R}] = [\mathbf{R}_2]\,[\mathbf{R}_1]$$

- Equating elements in the general expression gives nine equations in these eight unknowns.

$$u_{1_x},\ u_{1_y},\ u_{1_z},\ \square_1$$
$$u_{2_x},\ u_{2_y},\ u_{2_z},\ \square_2$$

Resolution of Rotations:

- We also have some additional constraint equations because the rotation axes are specified by unit vectors:

$$u_{1_x}^2 + u_{1_y}^2 + u_{1_z}^2 = 1$$
$$u_{2_x}^2 + u_{1_y}^2 + u_{2_z}^2 = 1$$
$$(Unit\ Vectors)$$

Resolution of Rotations:

- These two relationships reduce the system from eight down to six unknowns.
- (The third element of each unit vector is determined once we know the other two.)

$$u_{1x}^2 + u_{1y}^2 + u_{1z}^2 = 1$$
$$u_{2x}^2 + u_{1y}^2 + u_{2z}^2 = 1$$
$$(Unit\ Vectors)$$

Resolution of Rotations:

- We also have three normalization equations:

$$|V_j| = \sum_{i=1}^{3} a_{ij}^2 = 1 \qquad j = 1, 2, 3$$

Resolution of Rotations:

- We also have three normalization equations:

$$\left|V_j\right| = \sum_{i=1}^{3} a_{ij}^2 = 1 \qquad j = 1,\ 2,\ 3$$

- (Each column of the rotation matrix is a unit vector)

Resolution of Rotations:

- In addition we have three orthogonality conditions:

$$\mathbf{V_1} \bullet \mathbf{V_2} = 0$$
$$\mathbf{V_2} \bullet \mathbf{V_3} = 0$$
$$\mathbf{V_3} \bullet \mathbf{V_1} = 0$$

- (the columns of the rotation matrix form a right-handed triple)

Resolution of Rotations:

- *Once all this is taken into account the system is reduced to three independent equations*
- *But we have six independent unknowns!*

What does it all mean??

Resolution of Rotations:

- *Once all this is taken into account the system is reduced to three independent equations*
- *But we have six independent unknowns!*

What does it all mean??

Resolution of Rotations:

- *It means there are an infinite number of ways we can resolve a rotation into a sequence of two rotations!*

Resolution of Rotations:

- *For instance:*
  - *We can rotate from the starting position to anyplace else we like*
    - *Pick* $[\mathbf{R}_1]$ (with a $\mathbf{u}_1$ and θ_1)
  - *We can then rotate from there to the known final position*
    - *Solve for* $[\mathbf{R}_2]$ (with a $\mathbf{u}_2$ and θ_2)

Resolution of Rotations:

- *Alternatively, you can prescribe some other combination of the elements of the unknowns*

$$u_{1_x},\ u_{1_y},\ u_{1_z},\ \theta_1$$
$$u_{2_x},\ u_{2_y},\ u_{2_z},\ \theta_2$$

Resolution of Rotations:

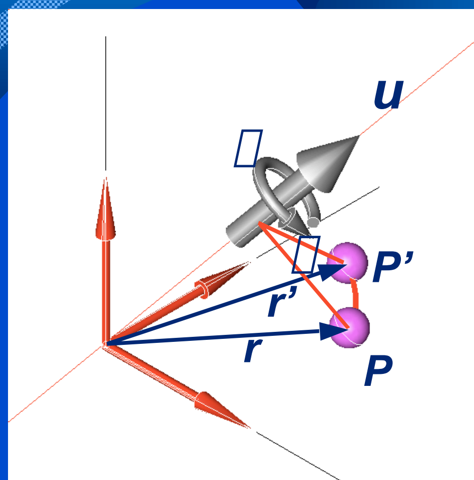
$$u_{1_x},\ u_{1_y},\ u_{1_z},\ \theta_1$$
$$u_{2_x},\ u_{2_y},\ u_{2_z},\ \theta_2$$

- *You can arbitrarily pick three of these scalar quantities (subject to a few obvious restrictions) and solve for the others*

What's *obvious*?

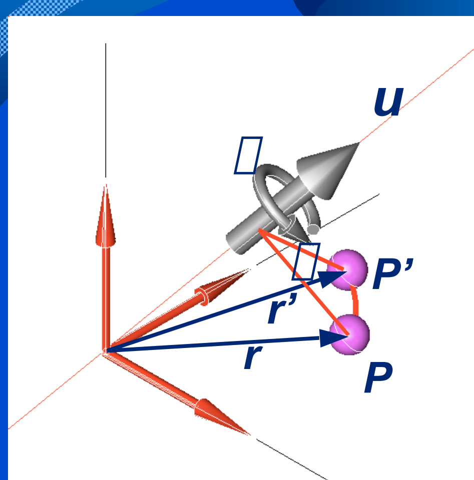
Obviously don't pick **u** bigger than 1 for example!

Derivation of the Velocity Matrix



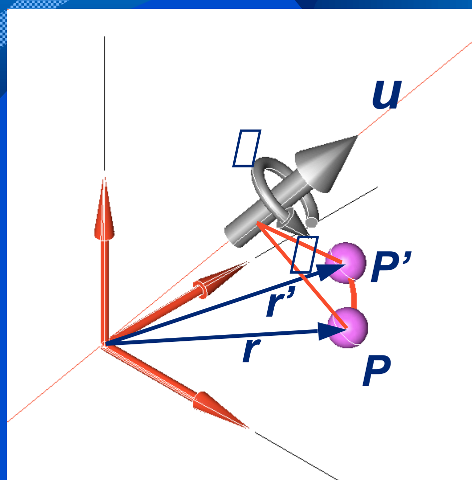
- *Suppose θ is time varying so $\theta = \theta(t)$*

Derivation of the Velocity Matrix



$$\boldsymbol{V} = \frac{d}{dt}\boldsymbol{r}' = \frac{d}{dt}[\mathbf{R}]\,\boldsymbol{r}$$

Derivation of the Velocity Matrix



$$V = \frac{d}{dt} r' = \frac{d}{dt} [\mathbf{R}]\, r$$

But **r** is a constant!

Derivation of the Velocity Matrix

$$V = \frac{d}{dt} r' = \frac{d}{dt} [\mathbf{R}]\, r$$

r is a constant, so you only need $\frac{d}{dt} [\mathbf{R}]$

Derivation of the Velocity Matrix

- *For infinitesimal rotations we can make small angle approximations in the general rotation matrix:*

$$cos\,(\square)\ \square\ 1$$
$$sin\,(\square)\ \square\ 0$$

Derivation of the Velocity Matrix

- *Then:*

$$\left[\mathbf{R}_{\mathbf{INF}}\right] = \begin{bmatrix} 1 & -u_z\square & u_y\square \\ u_z\square & 1 & -u_x\square \\ -u_y\square & u_x\square & 1 \end{bmatrix}$$

Derivation of the Velocity Matrix

- *And:*

$$\frac{d\left[\mathbf{R}_{\mathbf{INF}}\right]}{dt} = \begin{bmatrix} 0 & -u_z\square & u_y\square \\ u_z\square & 0 & -u_x\square \\ -u_y\square & u_x\square & 0 \end{bmatrix} = [\square]$$

Derivation of the Velocity Matrix

- *Define*

$$\square = \begin{bmatrix} u_x\square \\ u_y\square \\ u_z\square \end{bmatrix} = \begin{bmatrix} \square_x \\ \square_y \\ \square_z \end{bmatrix}$$

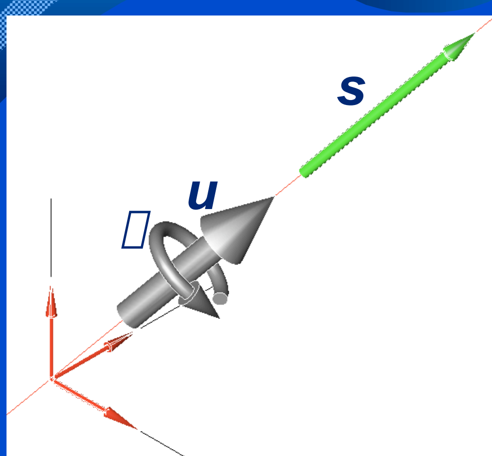
Derivation of the Velocity Matrix

- Then

$$\frac{d\left[\mathbf{R}_{\mathbf{INF}}\right]}{dt} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = [\Omega]$$

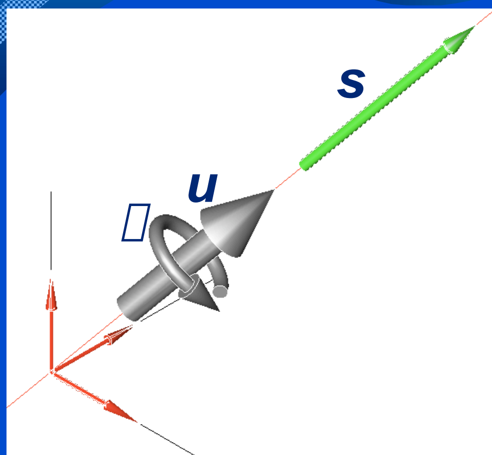
$$\boldsymbol{V} = [\Omega]\,\boldsymbol{r}$$

Screw Displacement About an Axis Through the Origin:



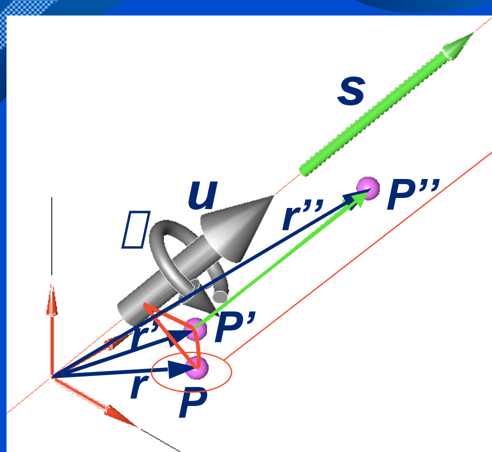
- A rotation plus a co-linear translation along the rotational axis is called a “Screw Displacement”

Screw Displacement About an Axis Through the Origin:



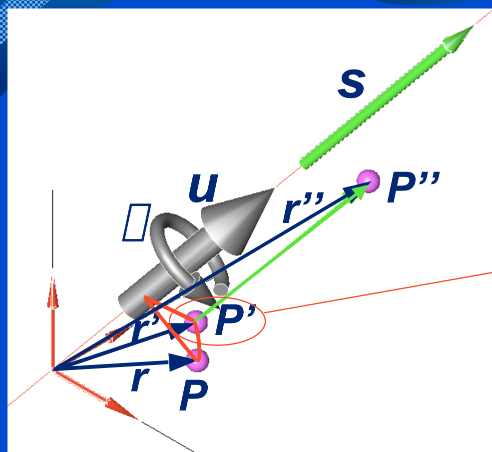
- For a screw motion the sequence of actions is unimportant
- The rotation can precede, follow, or coincide with the translation

Screw Displacement About an Axis Through the Origin:



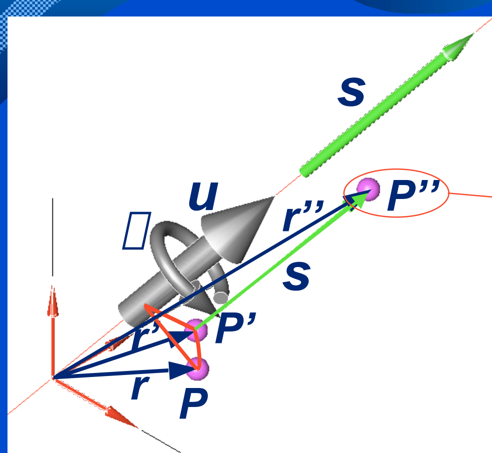
- Here, the particle P starts at the end of **r**

Screw Displacement About an Axis Through the Origin:



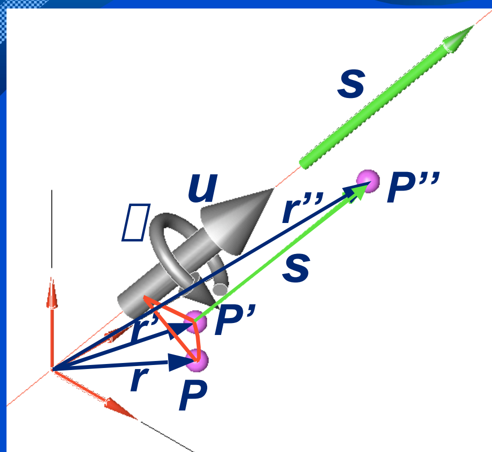
- *Here, the particle P starts at the end of* ***r***
- *After rotation it moves to P' located by the position vector* ***r'***

Screw Displacement About an Axis Through the Origin:



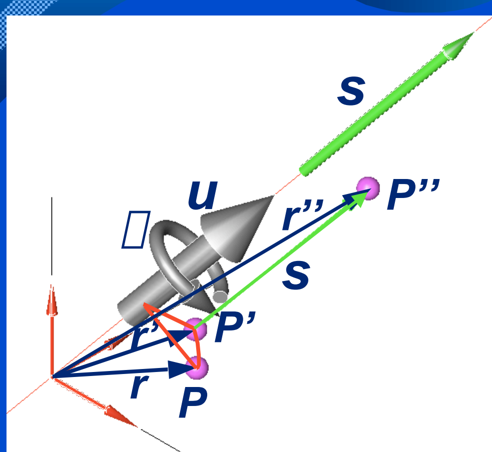
- *After the translation it moves on to P'' located by the position vector* ***r''***

Screw Displacement About an Axis Through the Origin:



- *Since the rotation and translation are about the same axes, the order is interchangeable!*

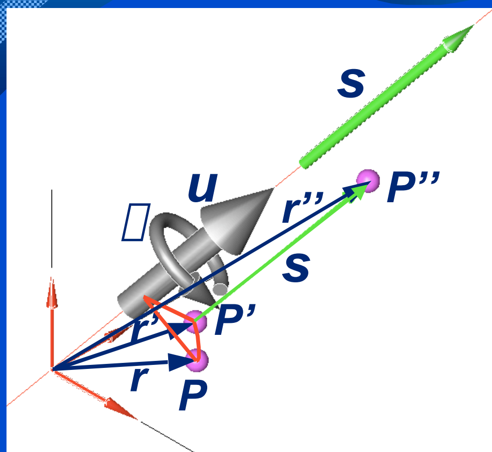
Screw Displacement About an Axis Through the Origin:



- *Mathematically, if the rotation takes place* *<u>before</u>* *the translation we have:*

$$\boldsymbol{r}' = [\mathrm{R}]\,\boldsymbol{r}$$
$$\boldsymbol{r}'' = \boldsymbol{r}' + \boldsymbol{s}$$
$$= [\mathrm{R}]\,\boldsymbol{r} + \boldsymbol{s}$$

Screw Displacement About an Axis Through the Origin:

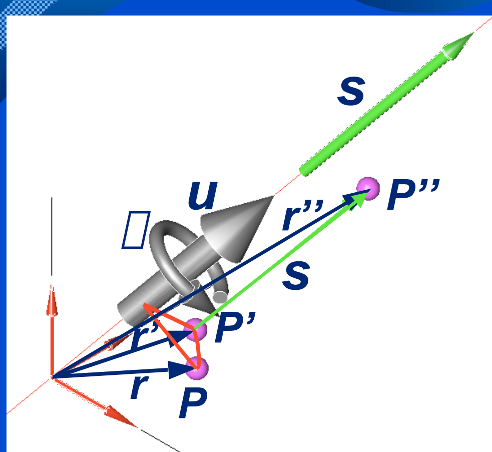


- *The sequence of operations is shown by the fact that the [R] operates on **r***

$$\boldsymbol{r}' = [\mathrm{R}]\,\boldsymbol{r}$$

$$\boldsymbol{r}'' = \boldsymbol{r}' + \boldsymbol{s} = [\mathrm{R}]\,\boldsymbol{r} + \boldsymbol{s}$$

Screw Displacement About an Axis Through the Origin:

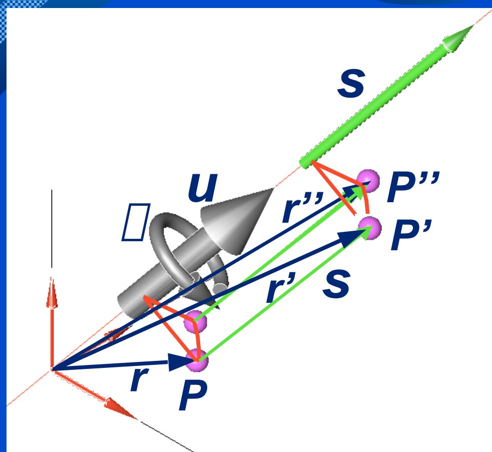


- *The rotation takes place first <u>even</u> if we write*

$$\boldsymbol{r}' = [\mathrm{R}]\,\boldsymbol{r}$$

$$\boldsymbol{r}'' = \boldsymbol{r}' + \boldsymbol{s} = [\mathrm{R}]\,\boldsymbol{r} + \boldsymbol{s} = \boldsymbol{s} + [\mathrm{R}]\,\boldsymbol{r}$$

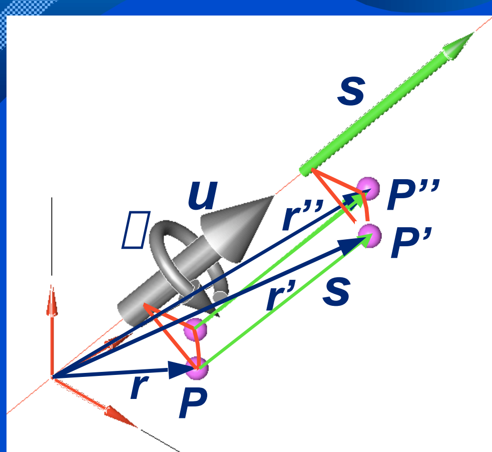
Screw Displacement About an Axis Through the Origin:



- *If the rotation takes place after the translation we have:*

$$r'' = [R]\{s + r\}$$

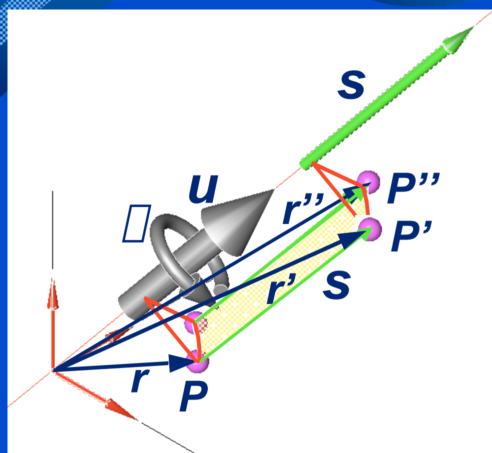
Screw Displacement About an Axis Through the Origin:



- *The sequence is shown by the fact that [R] operates on the displaced **r** after it has had **s** added to it*

$$r'' = [R]\{s + r\}$$

Screw Displacement About an Axis Through the Origin:

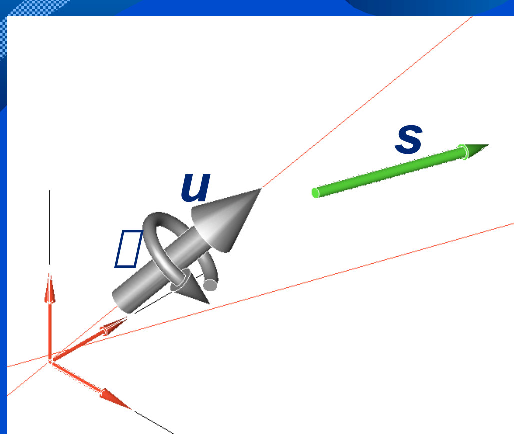


- *Notice we end up in the same place since we have done something akin to forming a parallelogram on the surface of a cylinder*

$$\begin{aligned} \boldsymbol{r}'' &= \boldsymbol{r}' + \boldsymbol{s} \\ &= [\mathrm{R}]\,\boldsymbol{r} + \boldsymbol{s} \\ &= \boldsymbol{s} + [\mathrm{R}]\,\boldsymbol{r} \\ &= [\mathrm{R}]\{\boldsymbol{s} + \boldsymbol{r}\} \end{aligned}$$

General Spatial Displacements

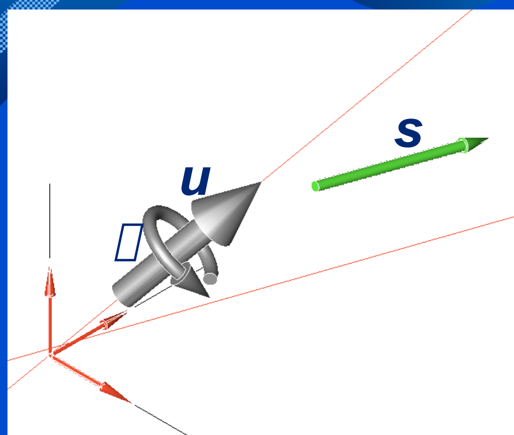
Axis Through the Origin



- *Suppose we have a pure rotation <u>followed</u> by a pure translation*

General Spatial Displacements

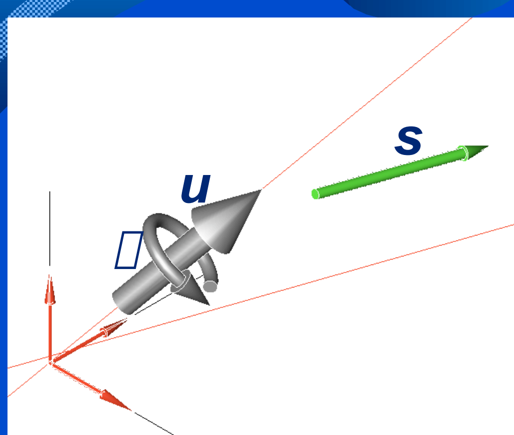
Axis Through the Origin



- Since the rotation and the translation are not about the same axes the order is <u>not</u> interchangeable

General Spatial Displacements

Axis Through the Origin, Rotation Followed by Translation

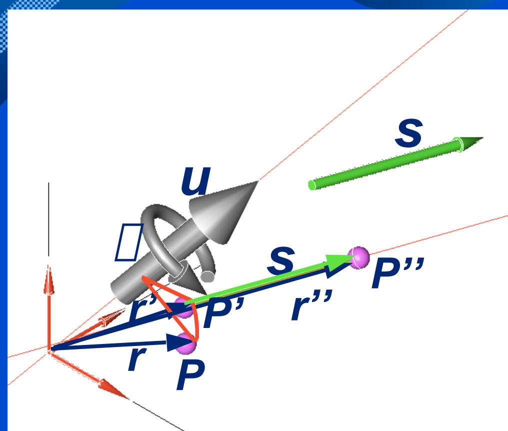


- The rotation is first since [R] operates on **r** and then **s** is added:

$$\boldsymbol{r}'' = [\mathrm{R}]\,\boldsymbol{r} + \boldsymbol{s}$$
$$= \boldsymbol{s} + [\mathrm{R}]\,\boldsymbol{r}$$

General Spatial Displacements

Axis Through the Origin, Rotation Followed by Translation

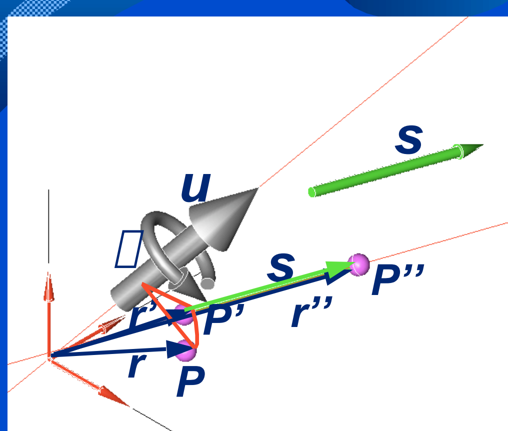


- *Here, rotation is first since [R] operates on **r** and then **s** is added:*

$$\mathbf{r}' = [\mathrm{R}]\,\mathbf{r} + \mathbf{s} = \mathbf{s} + [\mathrm{R}]\,\mathbf{r}$$

General Spatial Displacements

Axis Through the Origin, Rotation Followed by Translation



- *Notice that translation is a "free" vector*

General Spatial Displacements

Axis Through the Origin, Rotation Followed by Translation

$$[\mathrm{R}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Let

$$\boldsymbol{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

$$\boldsymbol{r}'' = \begin{bmatrix} r''_x \\ r''_y \\ r''_z \end{bmatrix}$$

General Spatial Displacements

Axis Through the Origin, Rotation Followed by Translation

Then

$$\begin{aligned} r''_x &= s_x + a_{11} r_x + a_{12} r_y + a_{13} r_z \\ r''_y &= s_y + a_{21} r_y + a_{22} r_y + a_{23} r_y \\ r''_z &= s_z + a_{31} r_z + a_{32} r_z + a_{33} r_z \end{aligned}$$

General Spatial Displacements

Axis Through the Origin, Rotation Followed by Translation

Suppose we introduce a new four-dimensional vector with no physical significance:

$$r''_w \equiv r_w + 0\, r_x + 0\, r_y + 0\, r_z$$

General Spatial Displacements

Axis Through the Origin, Rotation Followed by Translation

Then we can write our equations as:

$$\begin{aligned}
r''_w &\equiv r_w + 0\, r_x + 0\, r_y + 0\, r_z \\
r''_x &= s_x + a_{11}\, r_x + a_{12}\, r_y + a_{13}\, r_z \\
r''_y &= s_y + a_{21}\, r_y + a_{22}\, r_y + a_{23}\, r_y \\
r''_z &= s_z + a_{31}\, r_z + a_{32}\, r_z + a_{33}\, r_z
\end{aligned}$$

General Spatial Displacements

Axis Through the Origin, Rotation Followed by Translation

Also, by definition, let:

$$s_x \equiv p\, r_w$$
$$s_y \equiv q\, r_w$$
$$s_z \equiv t\, r_w$$

Here, p, q, and t are simply scalars

General Spatial Displacements

Axis Through the Origin, Rotation Followed by Translation

Then we can write:

$$\boldsymbol{r} \equiv \begin{bmatrix} r_w \\ \hdashline r_x \\ r_y \\ r_z \end{bmatrix} \qquad \boldsymbol{r}'' \equiv \begin{bmatrix} r''_w \\ \hdashline r''_x \\ r''_y \\ r''_z \end{bmatrix}$$

General Spatial Displacements: Rotation Followed by Translation

Now let

$$\mathbf{r}'' = [\mathbf{B}]\,\mathbf{r}$$

where [B] is a 4 X 4 matrix

General Spatial Displacements

Axis Through the Origin, Rotation Followed by Translation

Here,

$$[\mathbf{B}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ p & a_{11} & a_{12} & a_{13} \\ q & a_{21} & a_{22} & a_{23} \\ t & a_{31} & a_{32} & a_{33} \end{bmatrix}$$

General Spatial Displacements

Axis Through the Origin, Rotation Followed by Translation

So

$$\mathbf{r}'' = [\mathbf{B}]\,\mathbf{r}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ p & a_{11} & a_{12} & a_{13} \\ q & a_{21} & a_{22} & a_{23} \\ t & a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} r_w \\ r_x \\ r_y \\ r_z \end{bmatrix}$$

General Spatial Displacements

Axis Through the Origin, Rotation Followed by Translation

Now, let

$$r_w = 1$$

Then

$$p\,r_w = s_x$$

$$q\,r_w = s_y$$

$$t\,r_w = s_z$$

General Spatial Displacements

Axis Through the Origin, Rotation Followed by Translation

So

$$\boldsymbol{r} = \begin{bmatrix} 1 \\ \hline r_x \\ r_y \\ r_z \end{bmatrix}$$

General Spatial Displacements

Axis Through the Origin, Rotation Followed by Translation

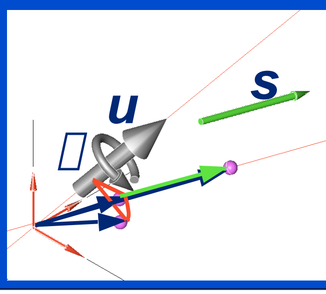
and

$$[\boldsymbol{B}] = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline s_x & a_{11} & a_{12} & a_{13} \\ s_y & a_{21} & a_{22} & a_{23} \\ s_z & a_{31} & a_{32} & a_{33} \end{array}\right] = \left[\begin{array}{c|c} \mathrm{I} & 0 \\ \hline \boldsymbol{s} & [R] \end{array}\right]$$

General Spatial Displacements

Axis Through the Origin, Rotation Followed by Translation

- *Keep in mind this restriction on what we have done:*
  - – ***U***’s axis passes through the origin!

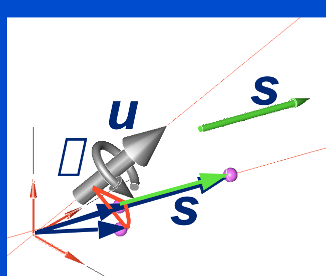


$$[B] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ s_x & a_{11} & a_{12} & a_{13} \\ s_y & a_{21} & a_{22} & a_{23} \\ s_z & a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} I & 0 \\ \mathbf{s} & [R] \end{bmatrix}$$

General Spatial Displacements

Axis Through the Origin, Rotation Followed by Translation

- *Keep in mind this restriction on what we have done:*
  - – ***U***’s axis passes through the origin!
- *Notice the translation is a “free” vector*



$$[B] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ s_x & a_{11} & a_{12} & a_{13} \\ s_y & a_{21} & a_{22} & a_{23} \\ s_z & a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} I & 0 \\ \mathbf{s} & [R] \end{bmatrix}$$

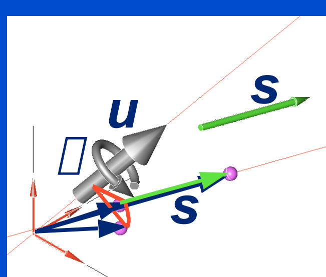
Homogeneous Coordinates

- *Previously, we needed to use matrix multiplication for rotations and matrix addition for translations*

$$\begin{aligned} \boldsymbol{r}'' &= [\mathrm{R}]\,\boldsymbol{r} + \boldsymbol{s} \\ &= \boldsymbol{s} + [\mathrm{R}]\,\boldsymbol{r} \end{aligned}$$

Homogeneous Coordinates

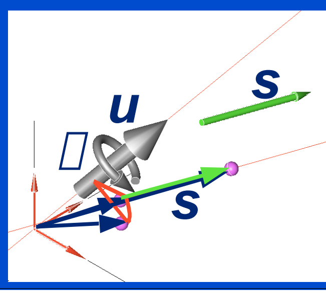
- *By introducing this new, fourth dimension, we are now able to combine translations and rotations into a single matrix multiplication*



$$[\boldsymbol{B}] = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline s_x & a_{11} & a_{12} & a_{13} \\ s_y & a_{21} & a_{22} & a_{23} \\ s_z & a_{31} & a_{32} & a_{33} \end{array}\right] = \left[\begin{array}{c|c} \mathrm{I} & 0 \\ \hline \boldsymbol{s} & [R] \end{array}\right]$$

Homogeneous Coordinates

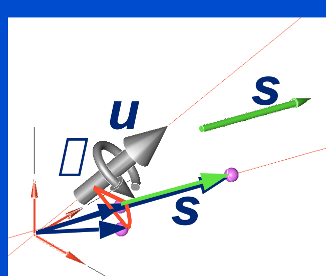
- *Using an n+1 dimensional space this way to represent an n-dimensional problem is an example of use of “homogeneous coordinates”*



$$[B] = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline s_x & a_{11} & a_{12} & a_{13} \\ s_y & a_{21} & a_{22} & a_{23} \\ s_z & a_{31} & a_{32} & a_{33} \end{array}\right] = \left[\begin{array}{c|c} \mathrm{I} & 0 \\ \hline \boldsymbol{s} & [R] \end{array}\right]$$

Homogeneous Coordinates

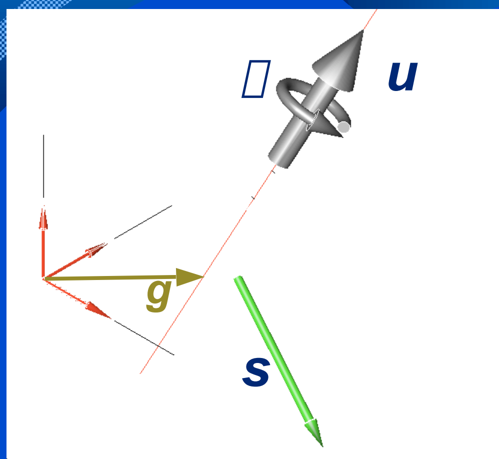
- *It allows us to treat both translations and rotations in a consistent, “homogeneous” manner*



$$[B] = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline s_x & a_{11} & a_{12} & a_{13} \\ s_y & a_{21} & a_{22} & a_{23} \\ s_z & a_{31} & a_{32} & a_{33} \end{array}\right] = \left[\begin{array}{c|c} \mathrm{I} & 0 \\ \hline \boldsymbol{s} & [R] \end{array}\right]$$

General Spatial Displacements

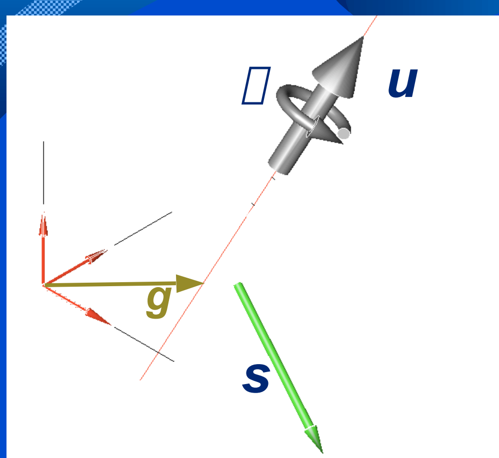
Axis NOT Through the Origin, Rotation Followed by Translation



- *Suppose the axis of rotation does not pass through the origin.*
- *It is offset from the origin by some vector **g**.*

General Spatial Displacements

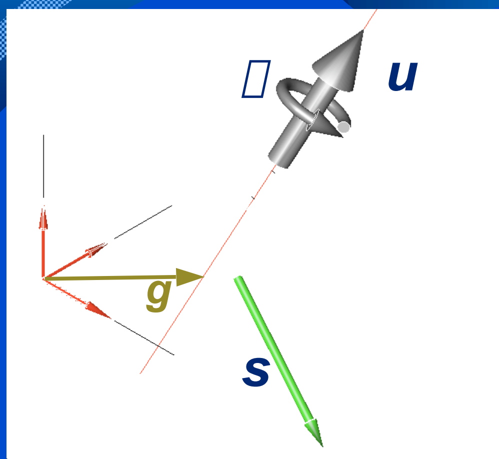
Axis NOT Through the Origin, Rotation Followed by Translation



- *Vector **g** goes to any point along the axis of rotation.*
- *It is not unique!*

General Spatial Displacements

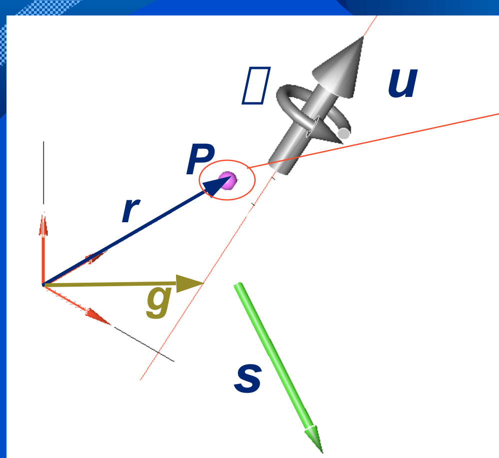
Axis NOT Through the Origin, Rotation Followed by Translation



- Again, the translation vector **s** is a free vector.
- It is not associated with any particular line of action.

General Spatial Displacements

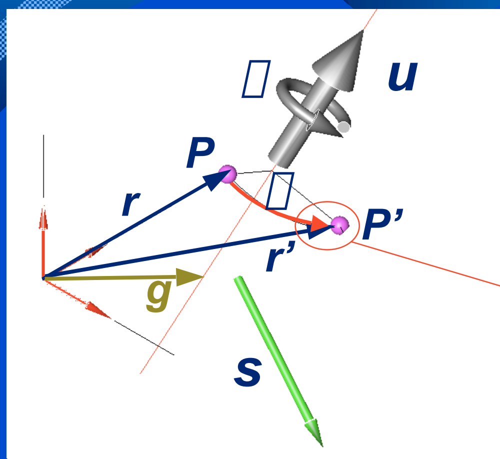
Axis NOT Through the Origin, Rotation Followed by Translation



- A particular particle P starts out at the end of a position vector **r**

General Spatial Displacements

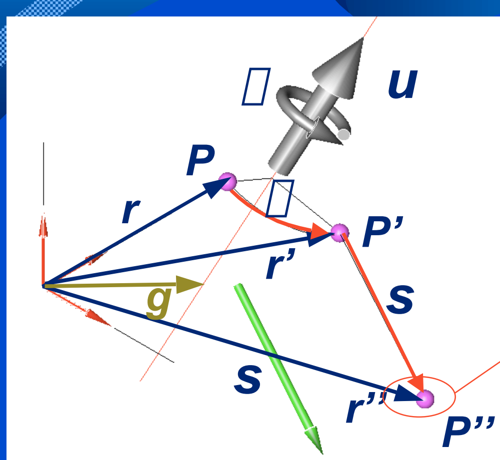
Axis NOT Through the Origin, Rotation Followed by Translation



- *P swings about **u** through angle θ.*
- *This brings it to position P' located by vector **r'***

General Spatial Displacements

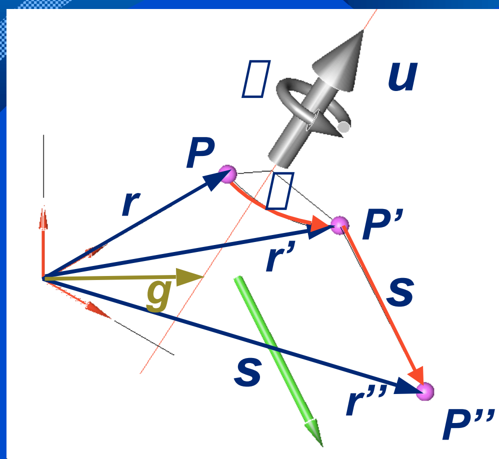
Axis NOT Through the Origin, Rotation Followed by Translation



- *P' then translates by vector s*
- *This brings it to its ending point, P'' located at **r''***

General Spatial Displacements

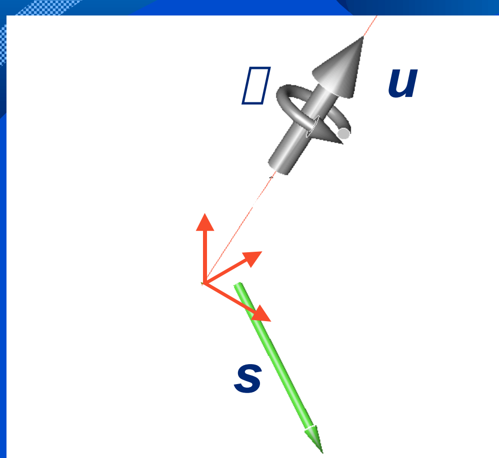
Axis NOT Through the Origin, Rotation Followed by Translation



- *How can we derive a matrix expression for this total displacement?*

General Spatial Displacements

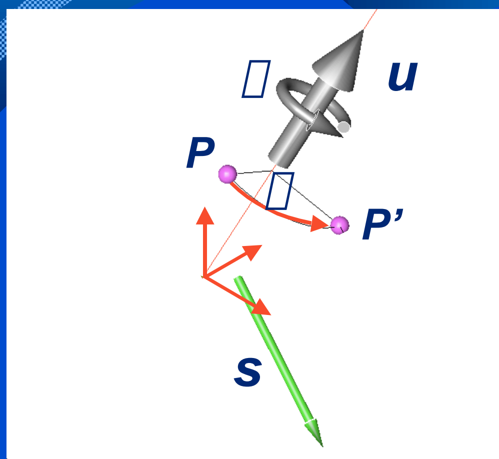
Axis NOT Through the Origin, Rotation Followed by Translation



- *To begin, let's temporarily shift our coordinate axes to the tip of **g***
- *This will bring the axis **u** through the new origin*

General Spatial Displacements

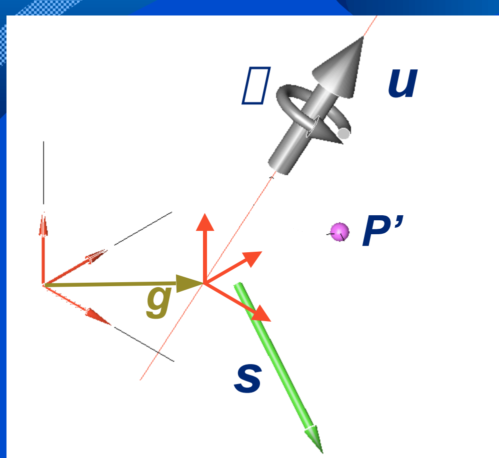
Axis NOT Through the Origin, Rotation Followed by Translation



- *The axis now passes through the new origin*
- *This makes it easy to rotate by angle □ about axis **u***

General Spatial Displacements

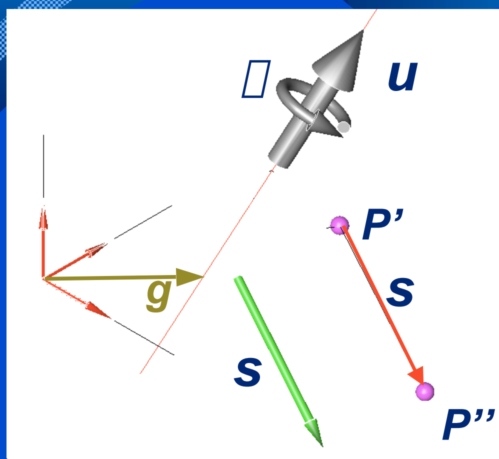
Axis NOT Through the Origin, Rotation Followed by Translation



- *With the rotation now done, we can then translate the axes back to where they belong*

General Spatial Displacements

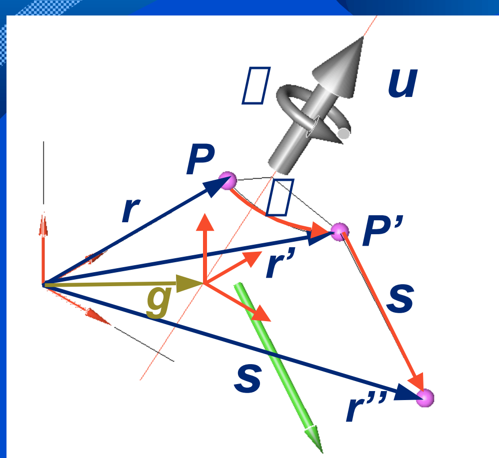
Axis NOT Through the Origin, Rotation Followed by Translation



- *Finally, we can apply the translation **s** bringing P’ to P’’.*

General Spatial Displacements

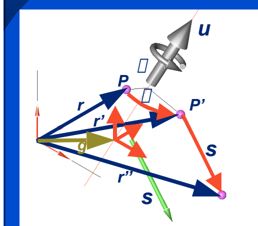
Axis NOT Through the Origin, Rotation Followed by Translation



- *Shifting the coordinate axes to the tip of **g** is mathematically the same as shifting the part backwards by **-g***

General Spatial Displacements

Axis NOT Through the Origin, Rotation Followed by Translation



- *So here's the overall sequence:*
  1. *Translate by -**g** to bring the axis through the origin*
  2. *Rotate about **u** by ▯*
  3. *Translate back by +**g***
  4. *Translate by **s***

General Spatial Displacements

Axis NOT Through the Origin, Rotation Followed by Translation

- *We can write out the steps this way:*
  1. *Translate by -**g** to bring the axis through the origin*

$$\{r - g\}$$

General Spatial Displacements

Axis NOT Through the Origin, Rotation Followed by Translation

- *We can write out the steps this way:*
  1. *Translate by -**g** to bring the axis through the origin*
  2. *Rotate about **u** by □*

$$[\mathrm{R}]\{r - g\}$$

General Spatial Displacements

Axis NOT Through the Origin, Rotation Followed by Translation

- *We can write out the steps this way:*
  1. *Translate by -**g** to bring the axis through the origin*
  2. *Rotate about **u** by □*
  3. *Translate back by +**g***

$$r' = [\mathrm{R}]\{r - g\} + g$$

General Spatial Displacements

Axis NOT Through the Origin, Rotation Followed by Translation

- *We can write out the steps this way:*
  1. *Translate by -**g** to bring the axis through the origin*
  2. *Rotate about **u** by □*
  3. *Translate back by +**g***
  4. *Translate by **s***

$$r'' = [\mathrm{R}]\{r - g\} + g + s$$

General Spatial Displacements

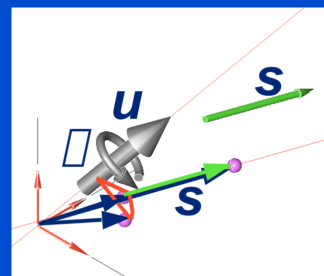
Axis NOT Through the Origin, Rotation Followed by Translation

- *How can we put all this into homogeneous coordinates?*

$$r'' = [\mathrm{R}]\{r - g\} + g + s$$

General Spatial Displacements

Axis NOT Through the Origin, Rotation Followed by Translation



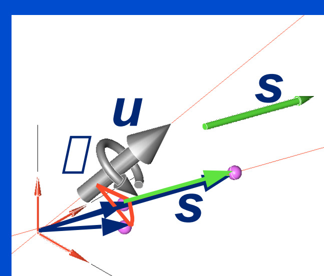
- *Recall that before, with the axis through the origin, we had* $\mathbf{r}'' = [\mathbf{B}]\,\mathbf{r}$

where

$$[B] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ s_x & a_{11} & a_{12} & a_{13} \\ s_y & a_{21} & a_{22} & a_{23} \\ s_z & a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \mathrm{I} & 0 \\ \mathbf{s} & [R] \end{bmatrix}$$

General Spatial Displacements

Axis NOT Through the Origin, Rotation Followed by Translation



- *This was for a rotation about the origin followed by a translation*

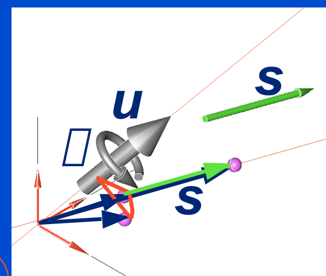
$$\mathbf{r}'' = [\mathbf{B}]\,\mathbf{r}$$

$$[B] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ s_x & a_{11} & a_{12} & a_{13} \\ s_y & a_{21} & a_{22} & a_{23} \\ s_z & a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \mathrm{I} & 0 \\ \mathbf{s} & [R] \end{bmatrix}$$

General Spatial Displacements

Axis NOT Through the Origin, Rotation Followed by Translation

- *The general form was*
  - *An identity submatrix*
  - *A zero submatrix*
  - *A translation submatrix*
  - *A rotation submatrix*



$$[B] = \begin{bmatrix} \mathrm{I} & 0 \\ \boldsymbol{s} & [R] \end{bmatrix}$$

General Spatial Displacements

Axis NOT Through the Origin, Rotation Followed by Translation

- *Back to the case of a rotation about an axis <u>not</u> through the origin, <u>followed</u> by a general translation*
- *The vector equation we derived for this was as follows:*

$$\boldsymbol{r}'' = [\mathrm{R}]\{\boldsymbol{r} - \boldsymbol{g}\} + \boldsymbol{g} + \boldsymbol{s}$$

General Spatial Displacements

Axis NOT Through the Origin, Rotation Followed by Translation

- *We want to put this series of vector calculations into a similar homogeneous form*

$$\boldsymbol{r}'' = [\mathrm{R}]\{\boldsymbol{r} - \boldsymbol{g}\} + \boldsymbol{g} + \boldsymbol{s}$$

General Spatial Displacements

Axis NOT Through the Origin, Rotation Followed by Translation

- *Start by expanding this out:*

$$\begin{aligned} \boldsymbol{r}'' &= [\mathrm{R}]\{\boldsymbol{r} - \boldsymbol{g}\} + \boldsymbol{g} + \boldsymbol{s} \\ &= \underbrace{[\mathrm{R}]\,\boldsymbol{r}}_{\text{Rotational Submatrix}} - \underbrace{[\mathrm{R} - \mathrm{I}]\,\boldsymbol{g} + \boldsymbol{s}}_{\text{Equivalent Translation}} \end{aligned}$$

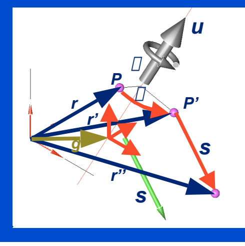
Rotational Submatrix

Equivalent Translation

General Spatial Displacements

Axis NOT Through the Origin, Rotation Followed by Translation

- *So, for a general rotation followed by a general translation:*



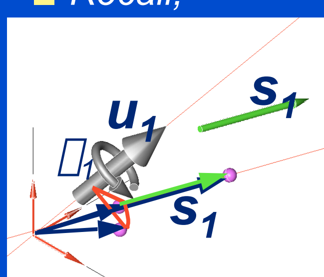
$$[B] = \left[\begin{array}{c|c} 1 & 0\ \ 0\ \ 0 \\ \hline \boldsymbol{s} - [R - I]\mathbf{g} & R \end{array}\right]$$

Equivalent Translation — Rotational Submatrix

Composition of General Spatial Displacements by 4X4 Matrices

Axis Through the Origin, Rotation Followed by Translation

- *Object:*
  – *to combine into a single matrix a displacement $[B_1]$ followed by a displacement $[B_2]$.*
- *Recall,*



$$[B] = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline s_x & a_{11} & a_{12} & a_{13} \\ s_y & a_{21} & a_{22} & a_{23} \\ s_z & a_{31} & a_{32} & a_{33} \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline \boldsymbol{s} & [R] \end{array}\right]$$

Composition of General Spatial Displacements by 4X4 Matrices

Axis Through the Origin, Rotation Followed by Translation

- After the first displacement we have:

$$\boldsymbol{r}' = \left[\mathrm{B}_1\right] \boldsymbol{r}$$

Composition of General Spatial Displacements by 4X4 Matrices

Axis Through the Origin, Rotation Followed by Translation

- After the second we have:

$$\boldsymbol{r}' = \left[\mathrm{B}_1\right] \boldsymbol{r}$$

$$\boldsymbol{r}'' = \left[\mathrm{B}_2\right] \boldsymbol{r}'$$

$$= \left[\mathrm{B}_2\right]\left[\mathrm{B}_1\right] \boldsymbol{r}$$

Composition of General Spatial Displacements by 4X4 Matrices

Axis Through the Origin, Rotation Followed by Translation

- *We want to merge these into one equivalent displacement so that:*

$$\boldsymbol{r}' = \left[\mathrm{B}_1\right] \boldsymbol{r}$$

$$\boldsymbol{r}'' = \left[\mathrm{B}_2\right] \boldsymbol{r}' = \left[\mathrm{B}_2\right]\left[\mathrm{B}_1\right] \boldsymbol{r} = \left[\mathrm{B}\right] \boldsymbol{r}$$

Composition of General Spatial Displacements by 4X4 Matrices

Axis Through the Origin, Rotation Followed by Translation

$$\left[B_1\right] = \left[\begin{array}{c|c} \boldsymbol{I} & \boldsymbol{0} \\ \hline \boldsymbol{s}_1 & \boldsymbol{R}_1 \end{array}\right] = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline s_{1_x} & & & \\ s_{1_y} & & R_1 & \\ s_{1_z} & & & \end{array}\right]$$

Composition of General Spatial Displacements by 4X4 Matrices

Axis Through the Origin, Rotation Followed by Translation

- Similarly:

$$\left[B_2\right] = \left[\begin{array}{c:c} I & 0 \\ \hdashline s_2 & R_2 \end{array}\right]$$

Composition of General Spatial Displacements by 4X4 Matrices

Axis Through the Origin, Rotation Followed by Translation

- Combining these we get:

$$\left[B\right] = \left[\begin{array}{c:c} I & 0 \\ \hdashline s & R \end{array}\right]$$

$$= \left[\begin{array}{c:c} I & 0 \\ \hdashline s_2 & R_2 \end{array}\right]\left[\begin{array}{c:c} I & 0 \\ \hdashline s_1 & R_1 \end{array}\right]$$

$$= \left[\begin{array}{c:c} I & 0 \\ \hdashline s_2 + \left[R_2\right] s_1 & \left[R_2\right]\left[R_1\right] \end{array}\right]$$

Composition of General Spatial Displacements by 4X4 Matrices

Axis Through the Origin, Rotation Followed by Translation

- Combining these we get:

$$[B] = \begin{bmatrix} I & 0 \\ s & R \end{bmatrix} = \begin{bmatrix} I & 0 \\ s_2 & R_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ s_1 & R_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ s_2 + [R_2]\, s_1 & [R_2][R_1] \end{bmatrix}$$

Resultant Translation

Resultant Rotation

Composition of General Spatial Displacements by 4X4 Matrices

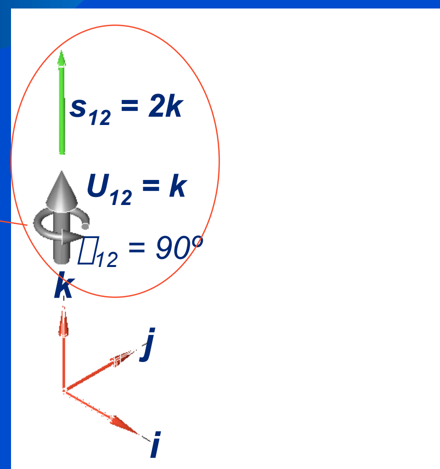
Axis Through the Origin, Rotation Followed by Translation

- In other words, the equivalent translation vector is:

$$\begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} = \begin{bmatrix} s_{2x} \\ s_{2y} \\ s_{2z} \end{bmatrix} + [R_2] \begin{bmatrix} s_{1x} \\ s_{1y} \\ s_{1z} \end{bmatrix}$$

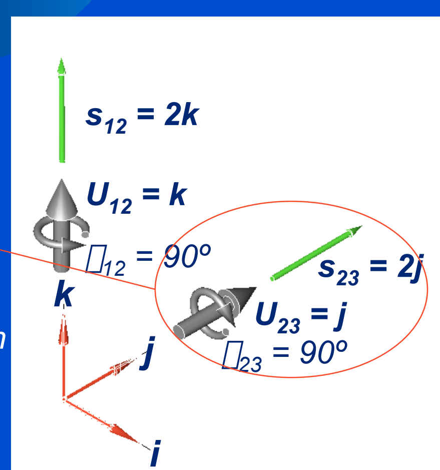
Screw Displacements: Example

- *Suppose we have a first screw displacement along the z axis*
  - *The rotation angle is 90º*
  - *The translation is 2 units*



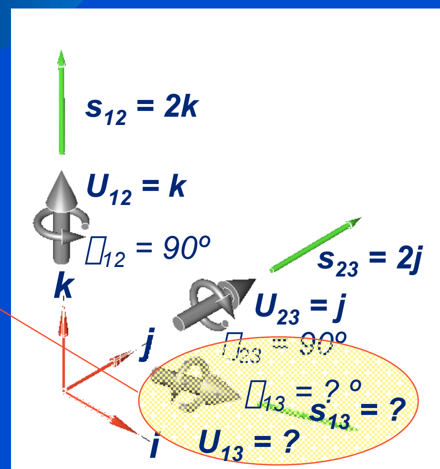
Screw Displacements: Example

- *This is followed by a second screw displacement along the y axis*
  - *Again, the rotation is 90º*
  - *The translation is again 2 units*



Screw Displacements: Example

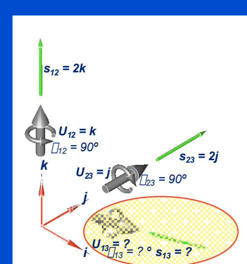
- What is the resultant screw?
  - Where is it located?
  - What is its line of action?
  - What is the equivalent single rotation angle?
  - What is the equivalent translation?



Screw Displacements: Example

$$[B_{13}] = [B_{23}]\,[B_{12}]$$

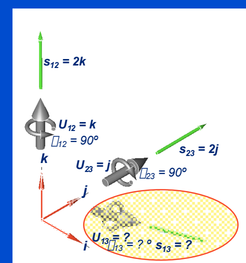
$$[B_{13}] = \begin{bmatrix} I & 0 \\ s_{13} & R_{13} \end{bmatrix}$$



Screw Displacements: Example

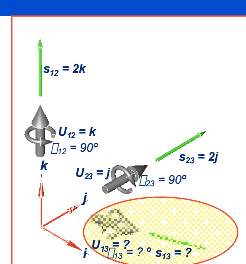
$$[B_{13}] = [B_{23}]\,[B_{12}]$$

$$[B_{13}] = \left[\begin{array}{c|c} I & 0 \\ \hline s_{13} & R_{13} \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline s_{23} & R_{23} \end{array}\right]\left[\begin{array}{c|c} I & 0 \\ \hline s_{12} & R_{12} \end{array}\right]$$



Screw Displacements: Example

$$[B_{13}] = \left[\begin{array}{c|c} I & 0 \\ \hline s_{13} & R_{13} \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline s_{23} & R_{23} \end{array}\right]\left[\begin{array}{c|c} I & 0 \\ \hline s_{12} & R_{12} \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline s_{23} + R_{23}\,s_{12} & R_{23}\;R_{12} \end{array}\right]$$



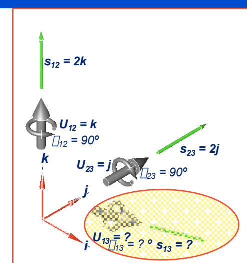
Screw Displacements: Example

$$[B_{13}] = \left[\begin{array}{c:c} I & 0 \\ \hdashline s_{13} & R_{13} \end{array}\right]$$

$$= \left[\begin{array}{c:c} I & 0 \\ \hdashline s_{23} & R_{23} \end{array}\right] \left[\begin{array}{c:c} I & 0 \\ \hdashline s_{12} & R_{12} \end{array}\right]$$

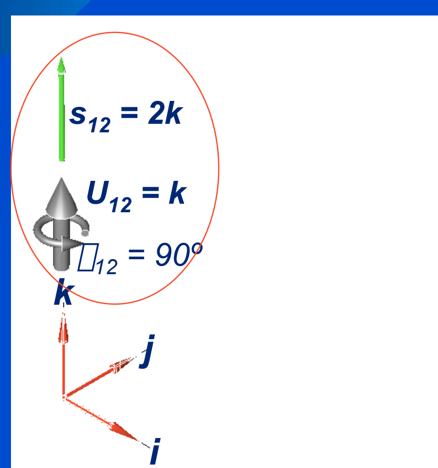
$$= \left[\begin{array}{c:c} I & 0 \\ \hdashline s_{23} + R_{23} s_{12} & R_{23} R_{12} \end{array}\right]$$

Equivalent s_{13} Equivalent $[R_{13}]$



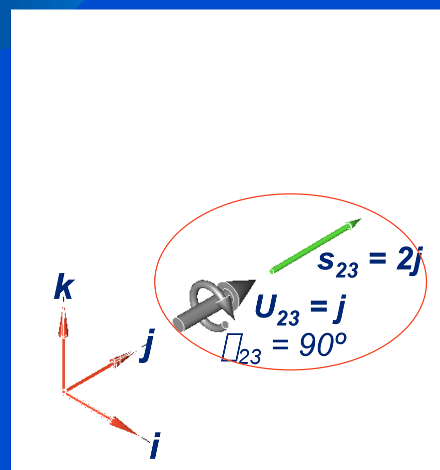
Screw Displacements: Example

$$[R_{12}] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Screw Displacements: Example

$$[R_{23}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

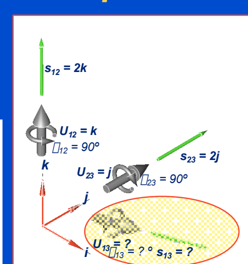


Screw Displacements: Example

$$[R_{13}] = [R_{23}][R_{12}]$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

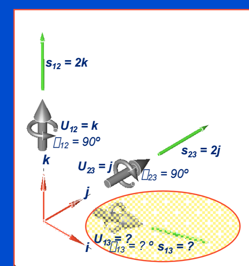
$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$



Screw Displacements: Example

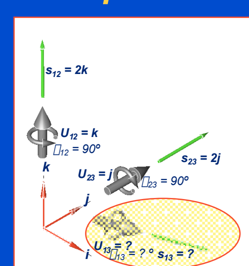
$$s_{12} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$s_{23} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$



Screw Displacements: Example

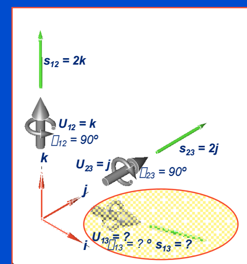
$$\begin{aligned} s_{13} &= s_{23} + \left[R_{23}\right] s_{12} \\ &= \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \end{aligned}$$



Screw Displacements: Example

- So $[B_{13}]$ becomes:

$$[B_{13}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

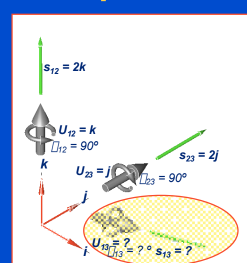


Screw Displacements: Example

- What does this all mean?

$$[B_{13}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

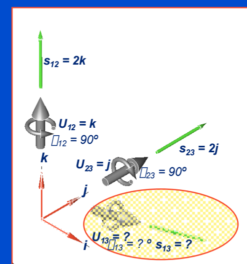
s_{13} $[R_{13}]$



Screw Displacements: Example

- Look at the columns of $[R_{13}]$:

$$[R_{13}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix}$$



Screw Displacements: Example

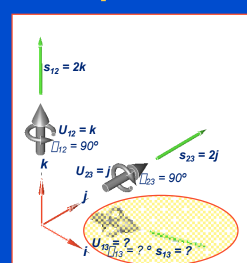
- Look at the columns of $[R_{13}]$:

i went to ***j***

j went to ***k***

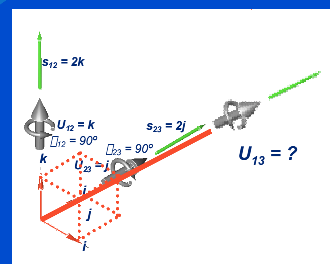
k went to ***i***

$$[R_{13}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix}$$



Screw Displacements: Example

- Thus this represents a rotation about a line along the diagonal of a cube



i went to ***j***

j went to ***k***

k went to ***i***

$$[R_{13}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix}$$

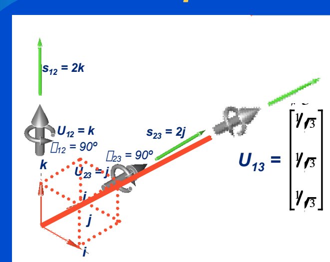
Screw Displacements: Example

- What is the $\boldsymbol{u}_{13}$?

$$|\boldsymbol{u}_{13}| = 1 = \sqrt{u_{13_x}^2 + u_{13_y}^2 + u_{13_z}^2}$$

$$u_{13_x}^2 = \frac{1}{3}$$

$$\boldsymbol{u}_{13} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k}) = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

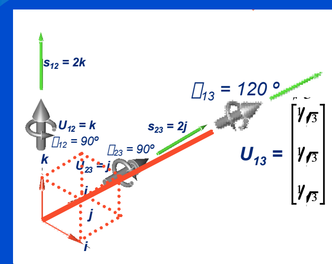


$$[R_{13}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix}$$

Screw Displacements: Example

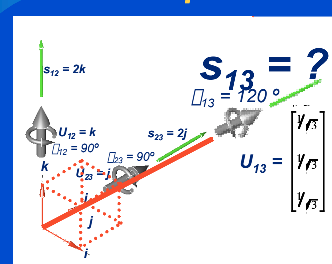
- *Because of symmetry, the angle of rotation $\square_{13}$ must be 120º*

$$[R_{13}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix}$$



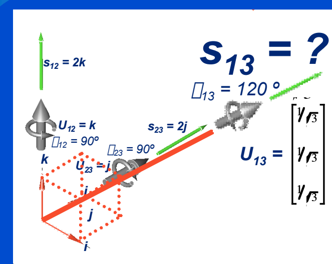
Screw Displacements: Example

- *What about the **s** ?*
- *Is this a screw motion?*



Screw Displacements: Example

- *This isn't a screw motion!*

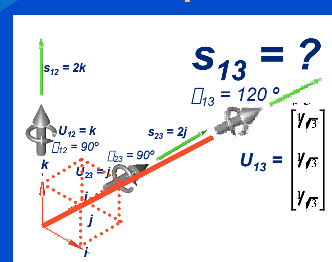


How does he know?

He don't say….

Screw Displacements: Example

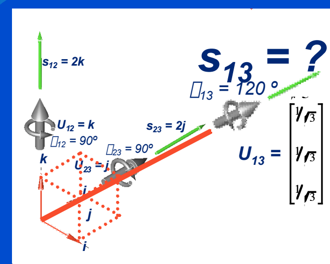
- *Obviously this isn't a screw motion, since you see the overall displacement is in the x-y plane and not along the* $\mathbf{u}_{13}$ *axis*



$$[B_{13}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

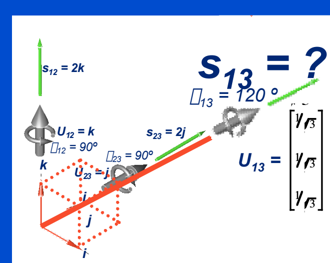
Screw Displacements: Example

- *How can we find the equivalent resultant screw for this case?*



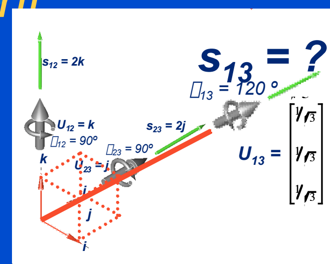
Equivalent Resultant Screw Displacement

- *Here's the general idea:*
  - *The rotation's the rotation's the rotation. It won't change.*



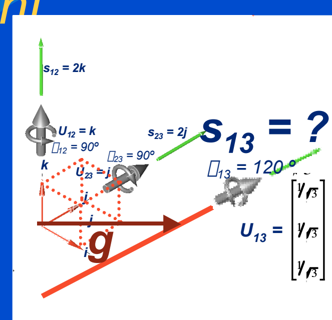
Equivalent Resultant Screw Displacement

- *Here's the general idea:*
  - *The rotation's the rotation's the rotation. It won't change.*
  - *The axis of rotation's the axis of rotation's the axis of rotation. It won't change either, when considered as a* ***free vector****!*



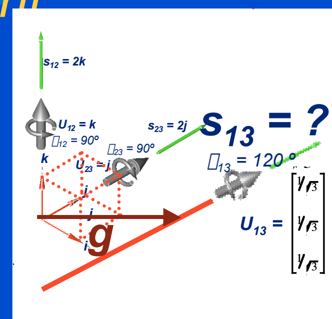
Equivalent Resultant Screw Displacement

- *So what must have changed?*
- *The location of the axis of rotation must have shifted away from the origin by some vector* ***g*** *so as to twist the* $\mathbf{s}_{13}$ *displacement vector into alignment with the rotation axis!*



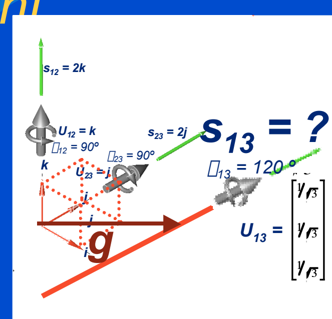
Equivalent Resultant Screw Displacement

- *Vector **g** can point to any point on the resultant screw axis*



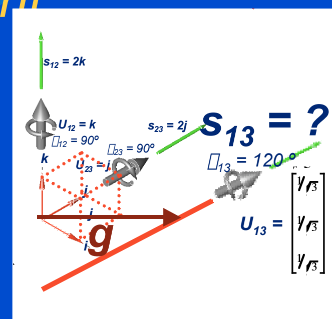
Equivalent Resultant Screw Displacement

- *What does the screw displacement we are looking for do to **<u>its own screw axis?</u>***



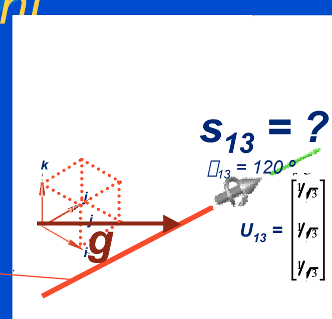
Equivalent Resultant Screw Displacement

- *Under the screw displacement, the screw axis itself would simply twist about its axis and shift along its own line of action.*



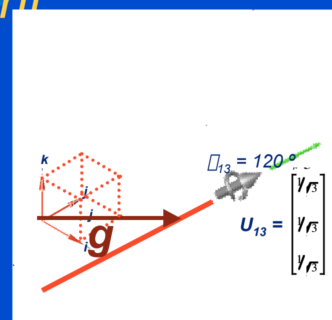
Equivalent Resultant Screw Displacement

- *So we are looking for the location of a line in the rigid body whose displacement under [B_{13}] is collinear with itself!*



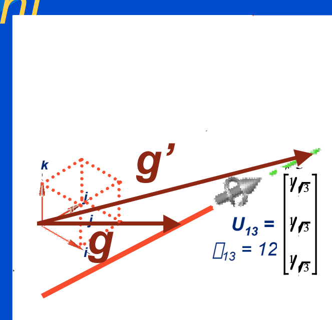
Equivalent Resultant Screw Displacement

- *Take any point on that unknown screw axis and call it **g***



Equivalent Resultant Screw Displacement

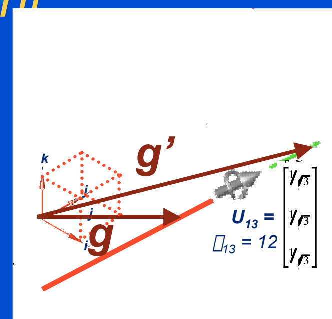
- *After the transformation $[B_{13}]$, **g** will have shifted to **g'**, also on the axis*



Equivalent Resultant Screw Displacement

- *After the transformation $[B_{13}]$, **g** will have shifted to **g’**, also on the axis*

$$\boldsymbol{g}' = \left[B_{13}\right] \boldsymbol{g}$$

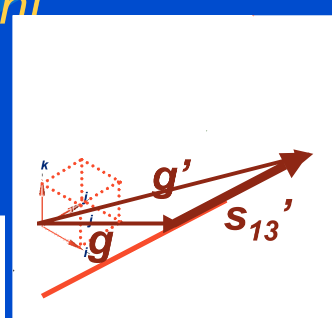


Equivalent Resultant Screw Displacement

- *After the transformation $[B_{13}]$, **g** will have shifted to **g’**, also on the axis*

$$\boldsymbol{g}' = \left[B_{13}\right] \boldsymbol{g}$$

$$\left[B_{13} - I\right] \boldsymbol{g} = \boldsymbol{s}_{13}'$$

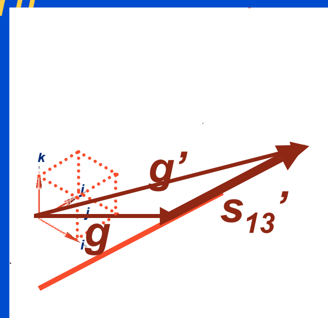


Equivalent Resultant Screw Displacement

- or

$$[B_{13} - I] = \left[\begin{array}{c:ccc} 0 & 0 & 0 & 0 \\ \hdashline 2 & -1 & 0 & 1 \\ 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]\begin{bmatrix} 1 \\ g_x \\ g_y \\ g_z \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2 - g_x + g_z \\ 2 + g_x - g_y \\ g_y - g_z \end{bmatrix}$$

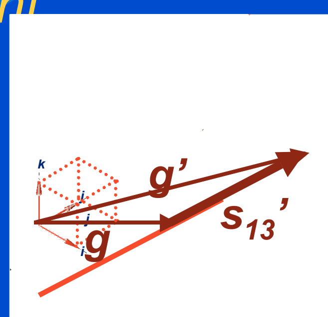


Equivalent Resultant Screw Displacement

- or

$$[B_{13} - I] = \left[\begin{array}{c:ccc} 0 & 0 & 0 & 0 \\ \hdashline 2 & -1 & 0 & 1 \\ 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]\begin{bmatrix} 1 \\ g_x \\ g_y \\ g_z \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2 - g_x + g_z \\ 2 + g_x - g_y \\ g_y - g_z \end{bmatrix}$$

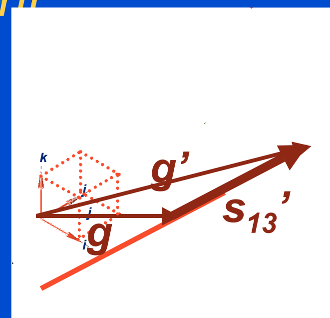


This vector must be in the direction of the screw axis, $\mathbf{u}_{13}$

Equivalent Resultant Screw Displacement

- So we can write three equations in four unknowns:

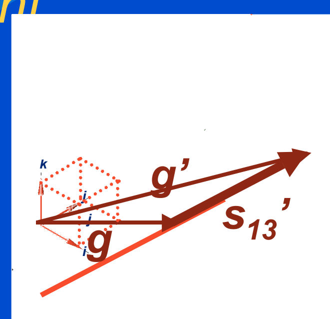
$$
\begin{aligned}
2 - g_x + g_z &= \lambda/\sqrt{3} \\
2 + g_x - g_y &= \lambda/\sqrt{3} \\
g_y - g_z &= \lambda/\sqrt{3}
\end{aligned}
$$



Equivalent Resultant Screw Displacement

- Lambda, here, is a scalar “stretch” factor that brings $\mathbf{s}_{13}$ to the unit length of the screw axis

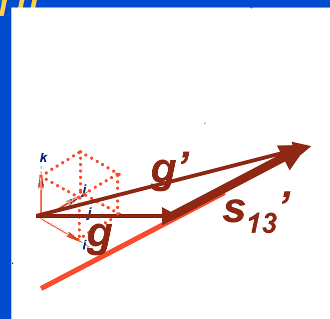
$$
\begin{aligned}
2 - g_x + g_z &= \lambda/\sqrt{3} \\
2 + g_x - g_y &= \lambda/\sqrt{3} \\
g_y - g_z &= \lambda/\sqrt{3}
\end{aligned}
$$



Equivalent Resultant Screw Displacement

- Since we have four unknowns and three equations, we can choose $g_z = 0$ arbitrarily.
- Then the first two equations gives us

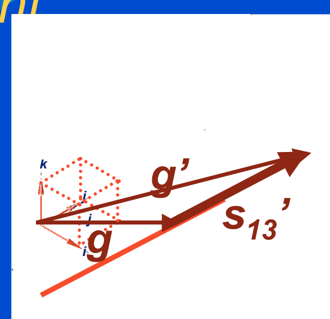
$$4 - g_y = 2\square/\sqrt{3}$$
$$g_y = \square/\sqrt{3}$$
$$\square = 4/\sqrt{3}$$



Equivalent Resultant Screw Displacement

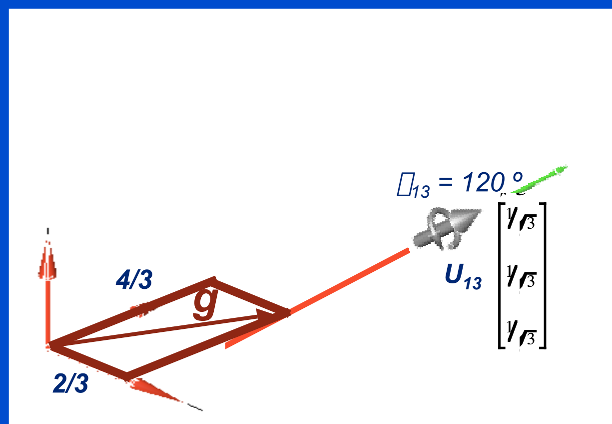
- So,

$$g_y = 4/3$$
$$g_x = 2/3$$

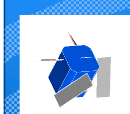


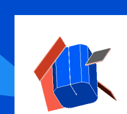
Equivalent Resultant Screw Displacement

- So the equivalent screw displacement then is

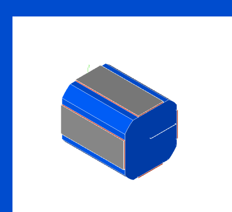


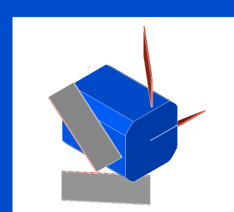
Project: Design of a Screw Mechanism for Solar Panel Deployment





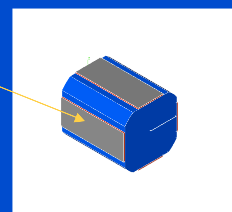
- A five foot long by five foot diameter satellite has four solar panels stowed flat along four flattened sides for launch.
- In orbit, these panels are to be deployed at a 40º angle to the axis of the satellite as shown.

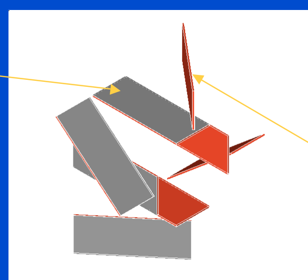


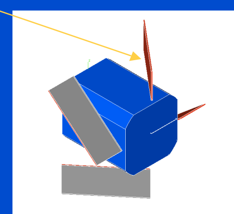


Project: Design of a Screw Mechanism for Solar Panel Deployment

- *The active surface of the panels is shown in gray.*
- *The back side of each panel is shown in red.*

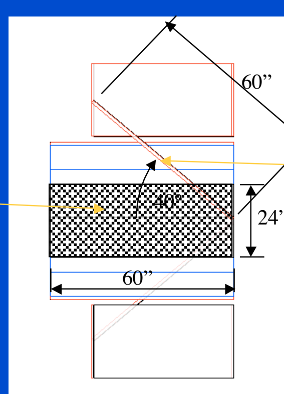






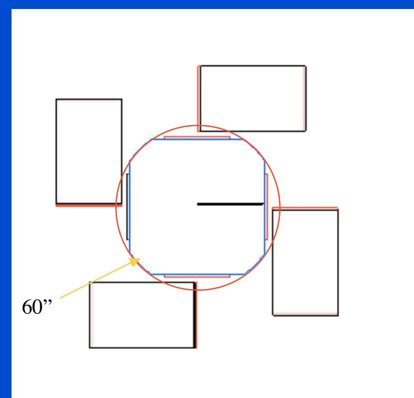
Project: Design of a Screw Mechanism for Solar Panel Deployment

- *Each panel is 24” by 60”.*



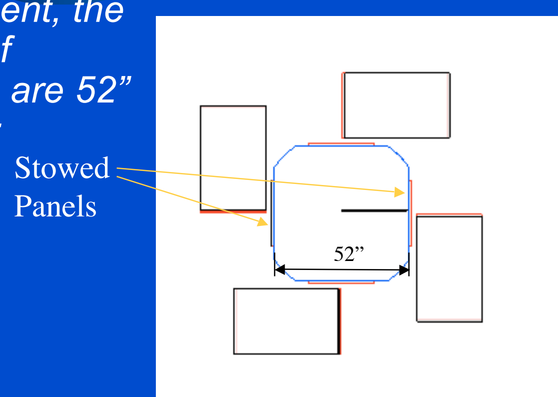
Project: Design of a Screw Mechanism for Solar Panel Deployment

- *The basic diameter of the satellite is 60”.*
- *It has been flattened on the four sides for mounting the panels.*



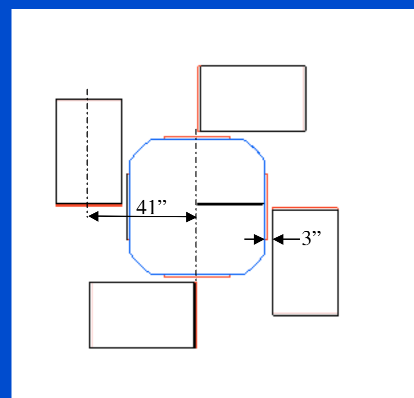
Project: Design of a Screw Mechanism for Solar Panel Deployment

- *Before deployment, the back surfaces of opposite panels are 52” apart as shown:*

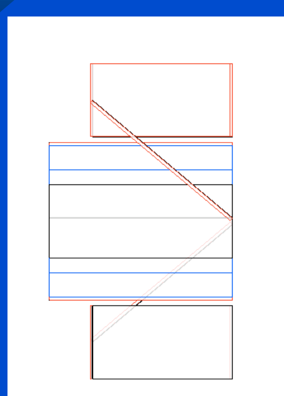


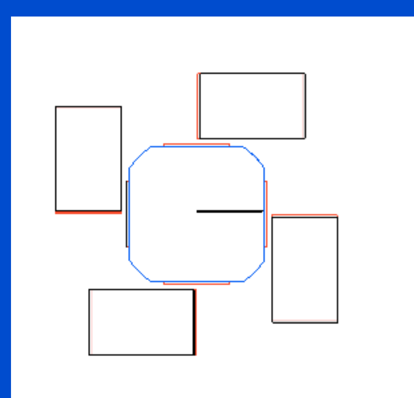
Project: Design of a Screw Mechanism for Solar Panel Deployment

- After deployment, the center line of a typical panel is to be located 41" from the center line of the satellite.
- This gives 3" clearance from the flattened side of the satellite.



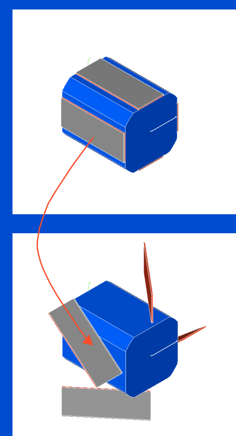
Project: Design of a Screw Mechanism for Solar Panel Deployment





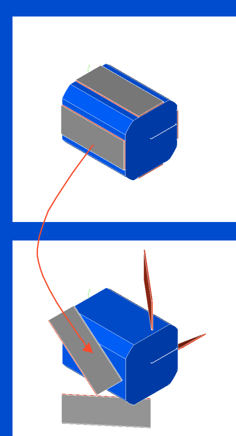
Project: Design of a Screw Mechanism for Solar Panel Deployment

- Task: Design a constant pitch screw mechanism to deploy a panel.
- Determine if part of the panel will interfere with the satellite body during this motion.
- If so, determine the interfering region and remove it.

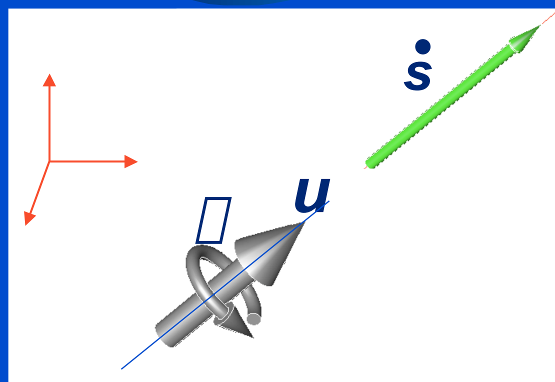


Project: Design of a Screw Mechanism for Solar Panel Deployment

- Hint:
  - Choose a convenient XYZ coordinate system.
  - Determine a succession of elementary displacements that will produce the desired motion.
  - Write matrices for these
  - Combine them to find the resultant screw
  - Draw up the final mechanism

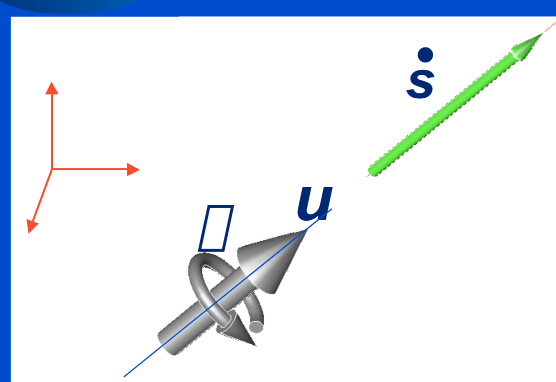


Instant Screws



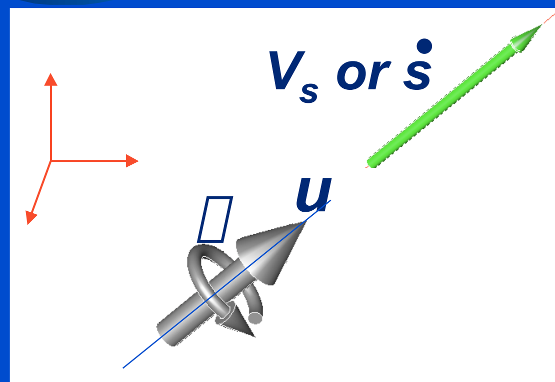
Instant Screws

- An Instant Screw for a motion is an axis about which the object is instantaneously rotating with an angular velocity ω
- At the same time, it is advancing along the **same** axis with a linear velocity $\dot{s}$



Instant Screws

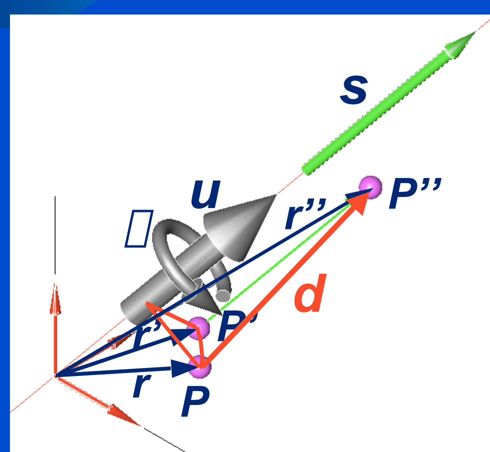
- The linear and angular velocities can be related to one another by the **lead** "*l*" of the screw.
- $V_s = l\,\omega\,u$



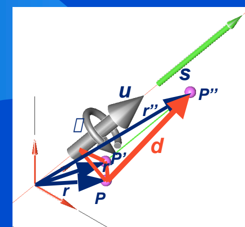
Instant Screws

- For a finite displacement we have:

$$\begin{aligned} d &= r'' - r \\ &= [B]\,r - r \\ &= [B - I]\,r \end{aligned}$$



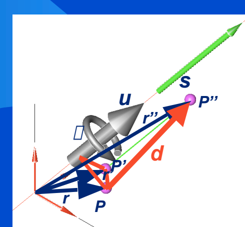
Instant Screws



$$[B - I] = \left[\begin{array}{c:c} I & 0 \\ \hdashline s & R \end{array}\right] - \left[\begin{array}{c:c} I & 0 \\ \hdashline 0 & I \end{array}\right]$$

$$= \left[\begin{array}{c:c} 0 & 0 \\ \hdashline s & R - I \end{array}\right]$$

Instant Screws

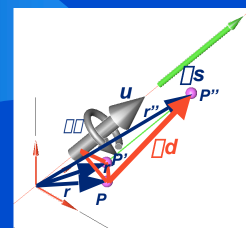


$$[B - I] =$$

$$\left[\begin{array}{c:ccc} 0 & 0 & 0 & 0 \\ \hdashline s_x & a_{11} - 1 & a_{12} & a_{13} \\ s_y & a_{21} & a_{22} - 1 & a_{23} \\ s_z & a_{31} & a_{32} & a_{33} - 1 \end{array}\right]$$

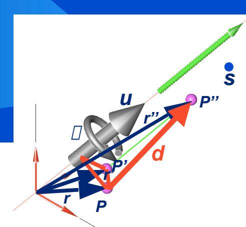
This is for finite displacements!

Instant Screws



$$V_{Average} = \left[\begin{array}{c:c} 0 & 0 \\ \hdashline \dfrac{\Delta s}{\Delta t} & \dfrac{R - I}{\Delta t} \end{array}\right] r$$

Instant Screws

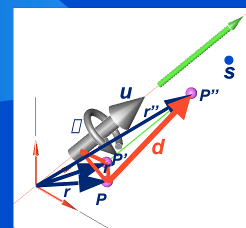


Let $\Delta t \rightarrow 0$

$d \rightarrow 0$

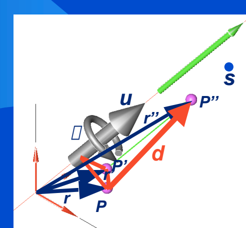
$$V = \left[\begin{array}{c:ccc} 0 & 0 & 0 & 0 \\ \hdashline & 0 & -\omega_z & \omega_y \\ \dot{s} & \omega_z & 0 & -\omega_x \\ & -\omega_y & \omega_x & 0 \end{array}\right] r$$

Instant Screws



$$V = \left[\begin{array}{c|c} 0 & 0 \\ \hline \dot{s} & \square \end{array}\right] r$$

Instant Screws



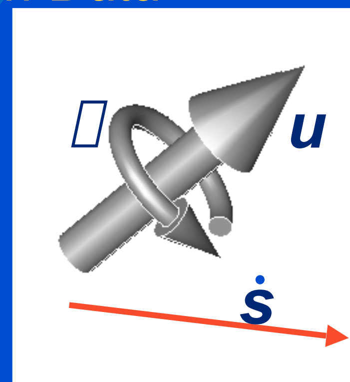
$$V = \left[\begin{array}{c|c} 0 & 0 \\ \hline \dot{s} & \square \end{array}\right] r$$

$$\dot{s} = \begin{bmatrix} \dot{s}_x \\ \dot{s}_y \\ \dot{s}_z \end{bmatrix}$$

$$r = \left[\begin{array}{c} 1 \\ \hline \mathrm{r_x} \\ \mathrm{r_y} \\ \mathrm{r_z} \end{array}\right]$$

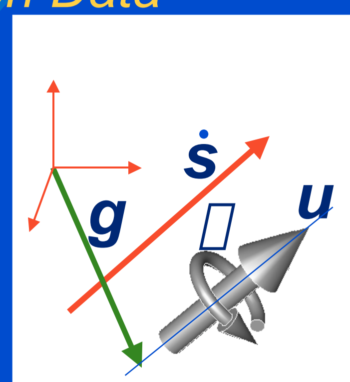
Determination of Instant Screw Axis From Given Data

- *Given an angular velocity specified by an axis vector **u** and by □*
- *And given a translational velocity specified by a vector ṡ (which may or may not lie parallel to **u**)*



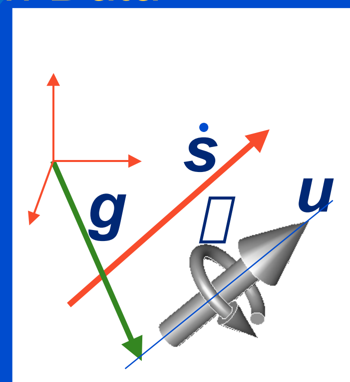
Determination of Instant Screw Axis From Given Data

- *Find the location of the line of action of the equivalent screw axis such that the corresponding translational velocity <u>WILL</u> lie along the axis **u***
- *Also find the lead of the screw so that* $\mathbf{V}_s$ *= l □ **u***



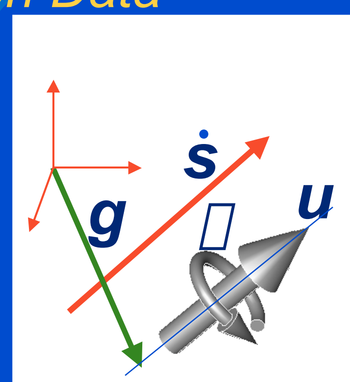
Determination of Instant Screw Axis From Given Data

- Let **g** be the unknown vector which locates the screw axis relative to our chosen coordinate system.



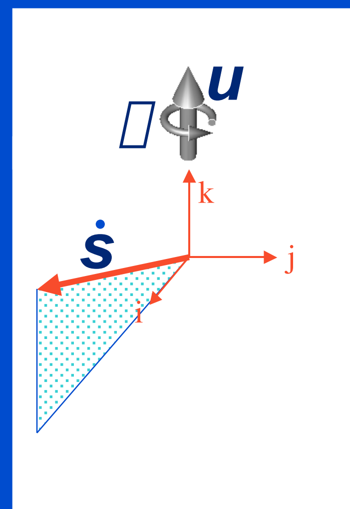
Determination of Instant Screw Axis From Given Data

- How can we find **g** ?
- We'll start by picking some more convenient coordinates in which to work.



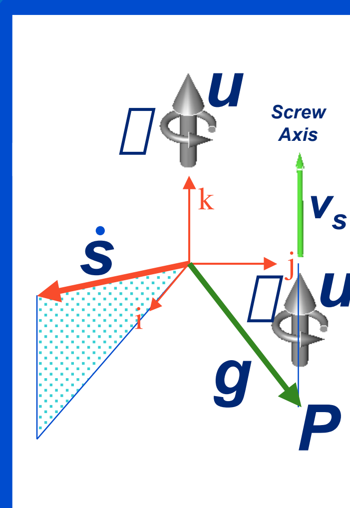
Procedure for Determination of Instant Screw Axis From Given Data

- *Step 1: Choose a coordinate system (or transform the coordinate system) so that the given axis vector is along the z axis. (i.e., **u** = **k**)*
- *Also choose the coordinates so that the given velocity vector lies in the x–z plane*



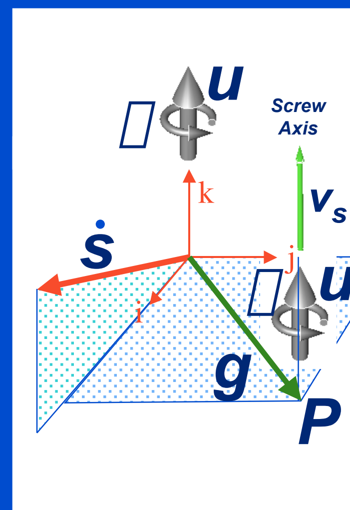
Procedure for Determination of Instant Screw Axis From Given Data

- *We are looking for the location of the screw axis relative to our chosen coordinate system.*
- *Let this be given by the unknown vector **g**.*
- ***g** specifies the position of a point P on the screw axis.*



Procedure for Determination of Instant Screw Axis From Given Data

- Since any point on the axis can be used to locate the instant screw choose the point where the axis cuts the x–y plane by letting $g_z = 0$.
- Thus, $\mathbf{g} = g_x \mathbf{i} + g_y \mathbf{j}$

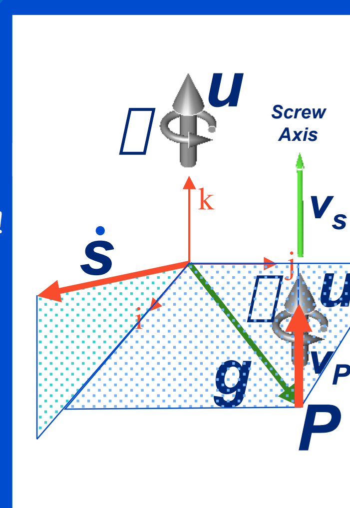


Procedure for Determination of Instant Screw Axis From Given Data

- What is the velocity of the point P on the screw axis?

The lead of the screw!

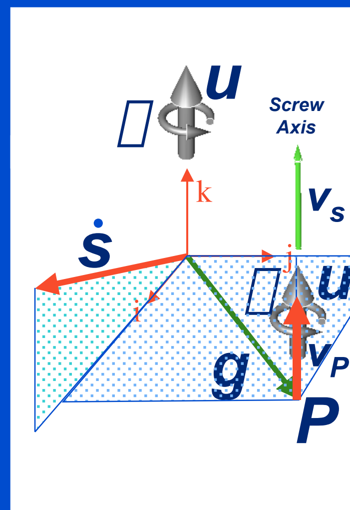
$$\boldsymbol{v}_P = \boldsymbol{l}\,\omega\,\boldsymbol{u}$$
$$= \dot{s} + \omega \times \boldsymbol{g}$$



Procedure for Determination of Instant Screw Axis From Given Data

- *Velocity of point P on the screw axis:*

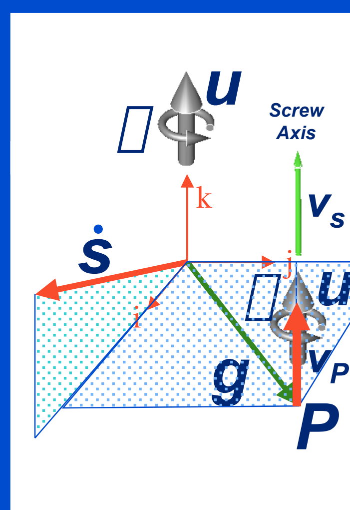
$$v_P = \dot{s} + [\square]\, g = \begin{bmatrix} 0 \\ 0 \\ l\,\square \end{bmatrix}$$



Procedure for Determination of Instant Screw Axis From Given Data

- *In this case, because of our choice of coordinates:*

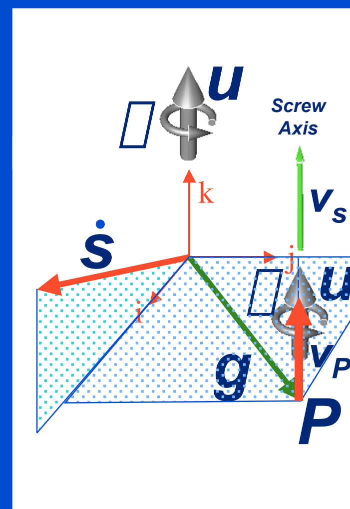
$$\dot{s} = \begin{bmatrix} \dot{s}_x \\ 0 \\ \dot{s}_z \end{bmatrix}$$



Procedure for Determination of Instant Screw Axis From Given Data

- *Also:*

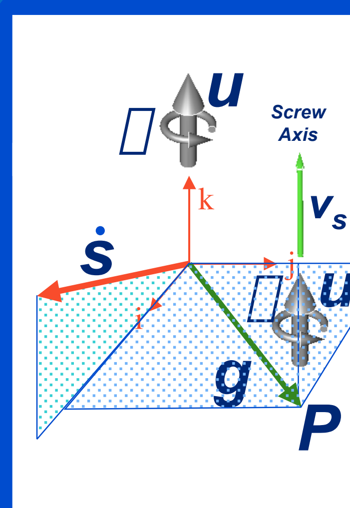
$$[\omega] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Procedure for Determination of Instant Screw Axis From Given Data

- *So:*

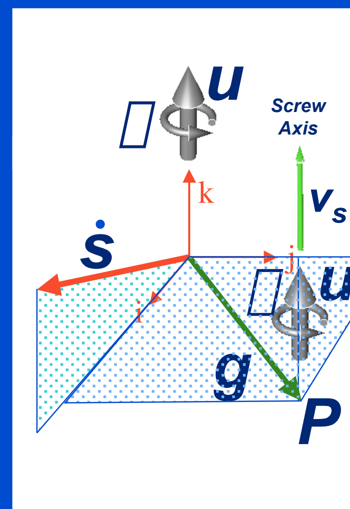
$$\begin{bmatrix} \dot{s}_x \\ 0 \\ \dot{s}_z \end{bmatrix} + \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} g_x \\ g_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ l\omega \end{bmatrix}$$



Procedure for Determination of Instant Screw Axis From Given Data

- This gives us three equations to work with:

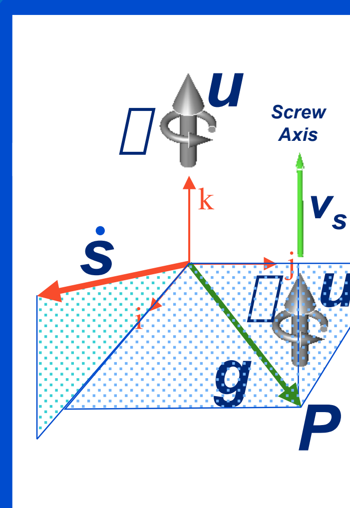
$$\dot{s}_x - \omega g_y = 0$$
$$\omega g_x = 0$$
$$\dot{s}_z = l\,\omega$$



Procedure for Determination of Instant Screw Axis From Given Data

- So:

$$g_x = 0$$
$$g_y = \frac{\dot{s}_x}{\omega}$$
$$g_z = 0$$
$$l = \frac{\dot{s}_z}{\omega}$$



Procedure for Determination of Instant Screw Axis From Given Data

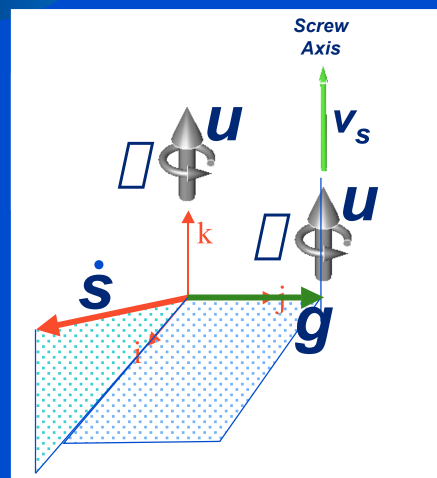
- *Thus, the solution is as shown:*

$$g_x = 0$$

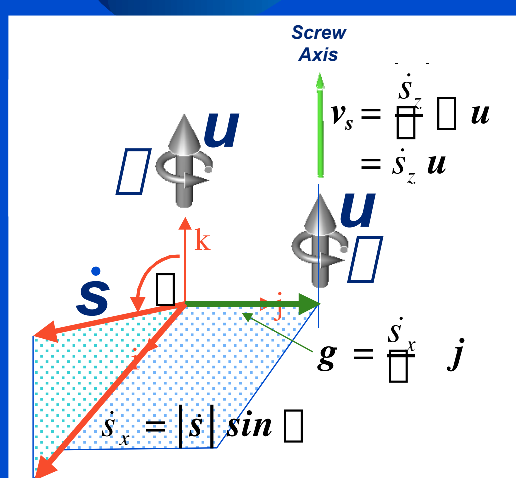
$$g_y = \frac{\dot{s}_x}{\square}$$

$$g_z = 0$$

$$l = \frac{\dot{s}_z}{\square}$$



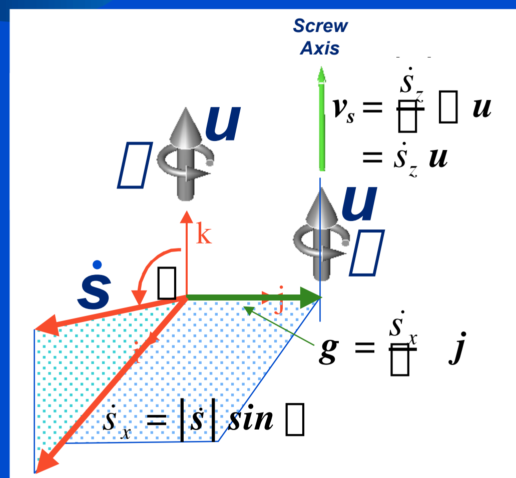
Procedure for Determination of Instant Screw Axis From Given Data



Procedure for Determination of Instant Screw Axis From Given Data

- We can see from the results that

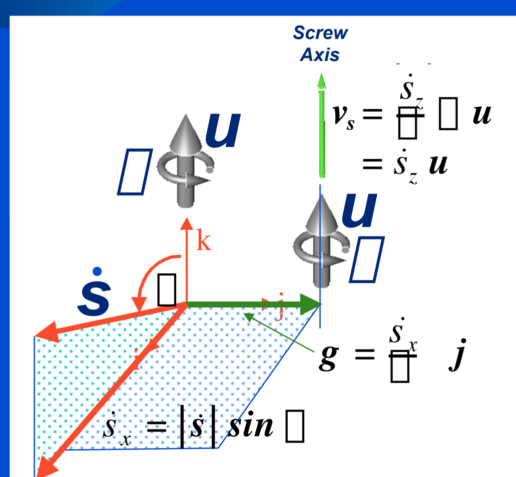
$$l\,\square = \dot{s}_z$$
$$= \dot{\mathbf{s}} \bullet \mathbf{u}$$
$$l = \frac{\dot{\mathbf{s}} \bullet \mathbf{u}}{\square}$$



Procedure for Determination of Instant Screw Axis From Given Data

- Also, we see that:

$$g = \frac{|u|\,|\dot{s}|\, sin\, \square}{\square}\, j$$
$$= \frac{u \,\square\, \dot{s}}{\square}$$

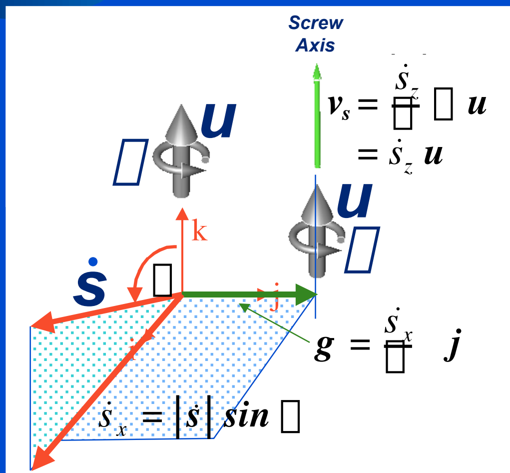


Procedure for Determination of Instant Screw Axis From Given Data

- *These two results are valid for any coordinate system!*

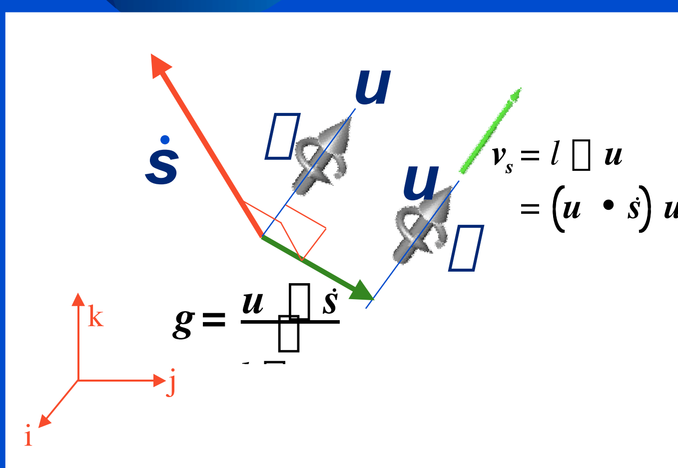
$$l = \frac{\dot{s} \bullet u}{\square}$$

$$g = \frac{|u||\dot{s}| \sin \square}{\square} j = \frac{u \square \dot{s}}{\square}$$



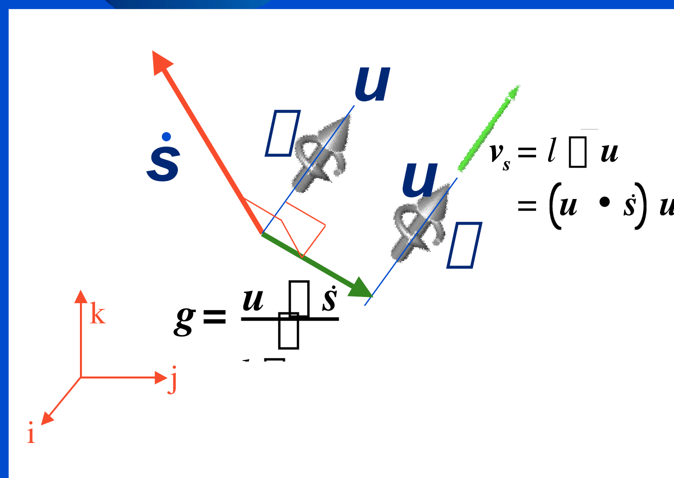
Instant Screw Axis From Given Data

- *In general:*

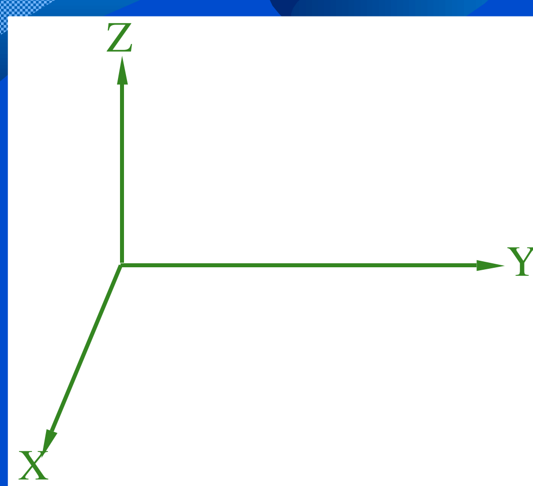


Instant Screw Axis From Given Data

- *Note: The velocity along the screw axis is independent of □ except through the lead of the screw.*

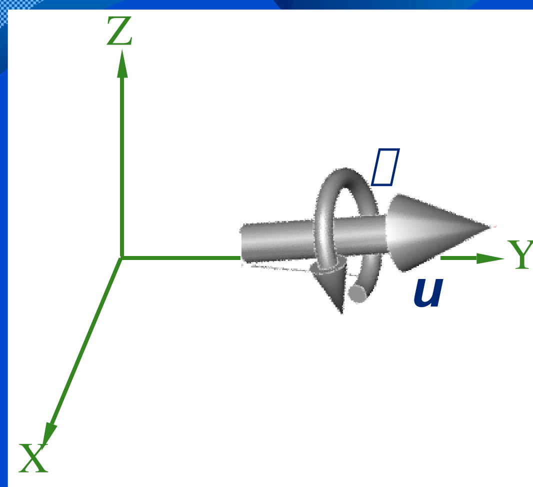


Coordinate Transformation for Pure Rotation About the Origin:



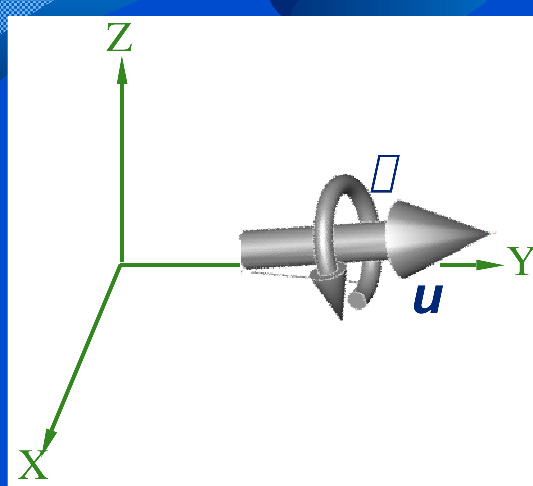
- *Suppose we are given the xyz starting coordinate system shown in green.*

Coordinate Transformation for Pure Rotation About the Origin:



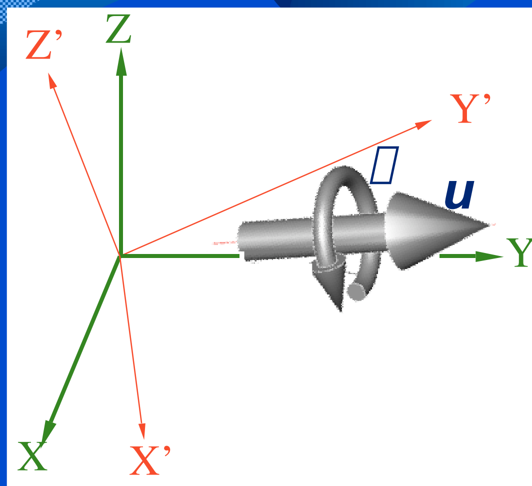
- We are also given the rotation axis vector ***u*** and angle *θ*
- Vector ***u*** passes through the origin.

Coordinate Transformation for Pure Rotation About the Origin:



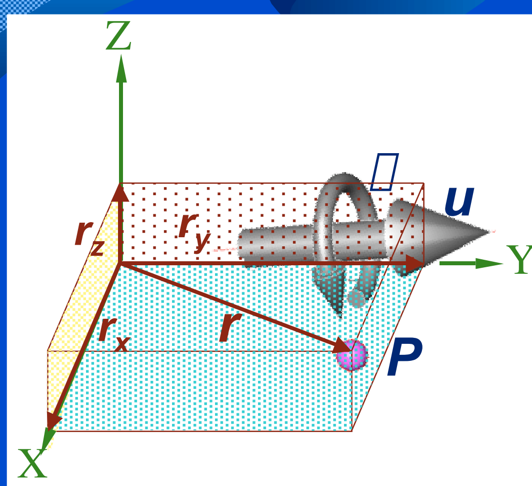
- We wish to rotate the coordinates by this ***u***, *θ*

Coordinate Transformation for Pure Rotation About the Origin:



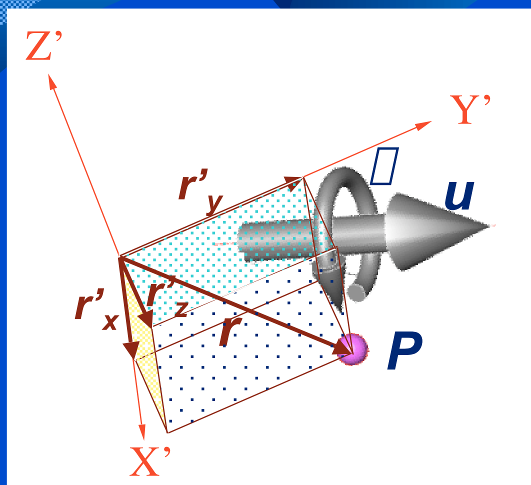
- *This will bring the coordinates to the position shown.*

Coordinate Transformation for Pure Rotation About the Origin:



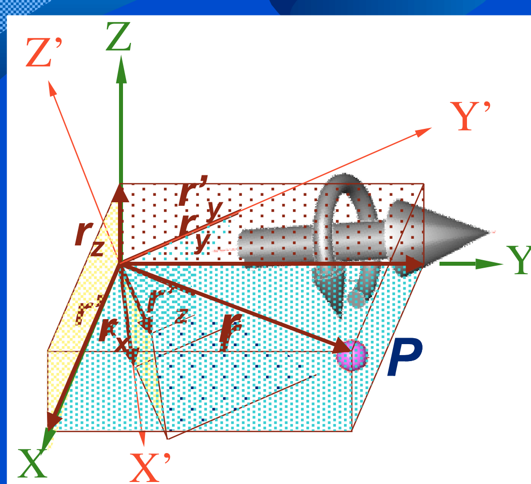
- *In the original coordinate system, a point P has components as shown:*

Coordinate Transformation for Pure Rotation About the Origin:



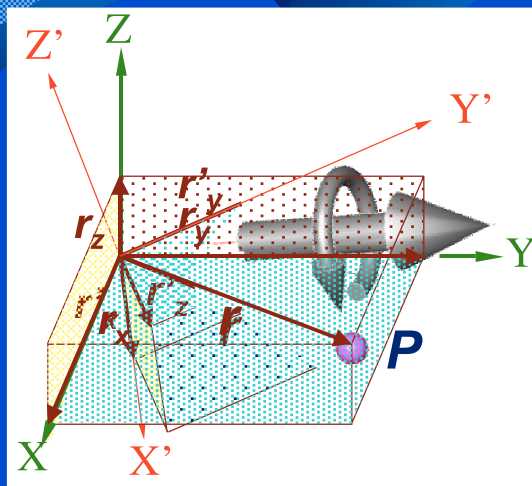
- In the new coordinate system, it has the components $\mathbf{r'}_x$, $\mathbf{r'}_y$, and $\mathbf{r'}_z$.

Coordinate Transformation for Pure Rotation About the Origin:



- We wish to determine a coordinate transformation matrix [T] that will leave the point P in its original position but rotate the coordinate system.

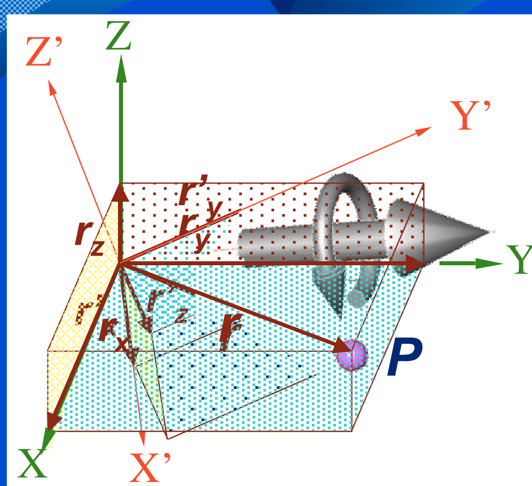
Coordinate Transformation for Pure Rotation About the Origin:



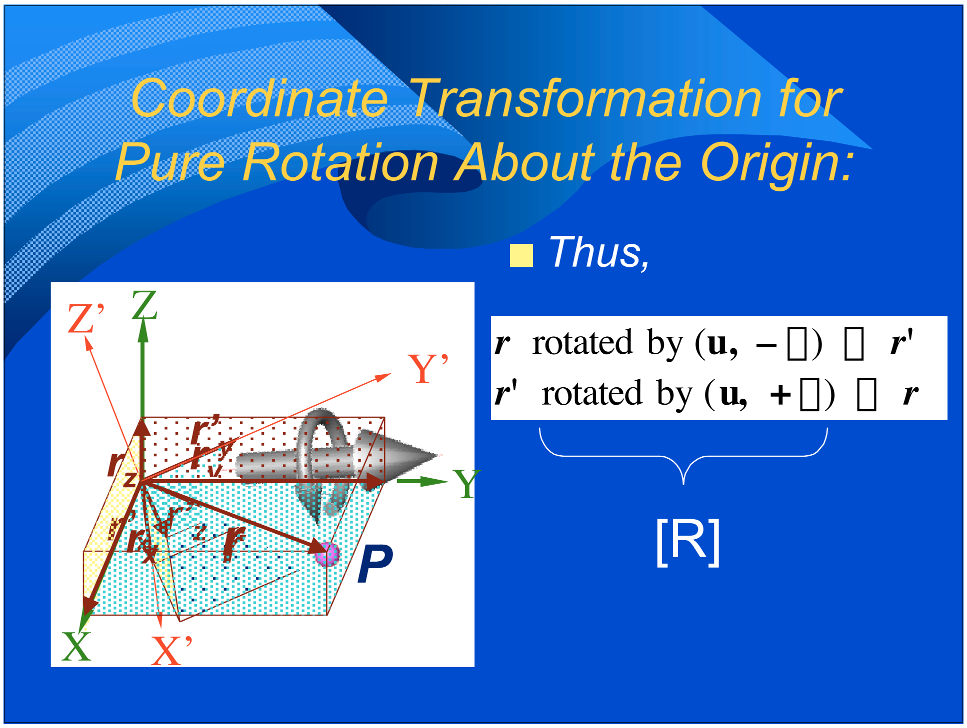
- So, we want to find a [T] such that

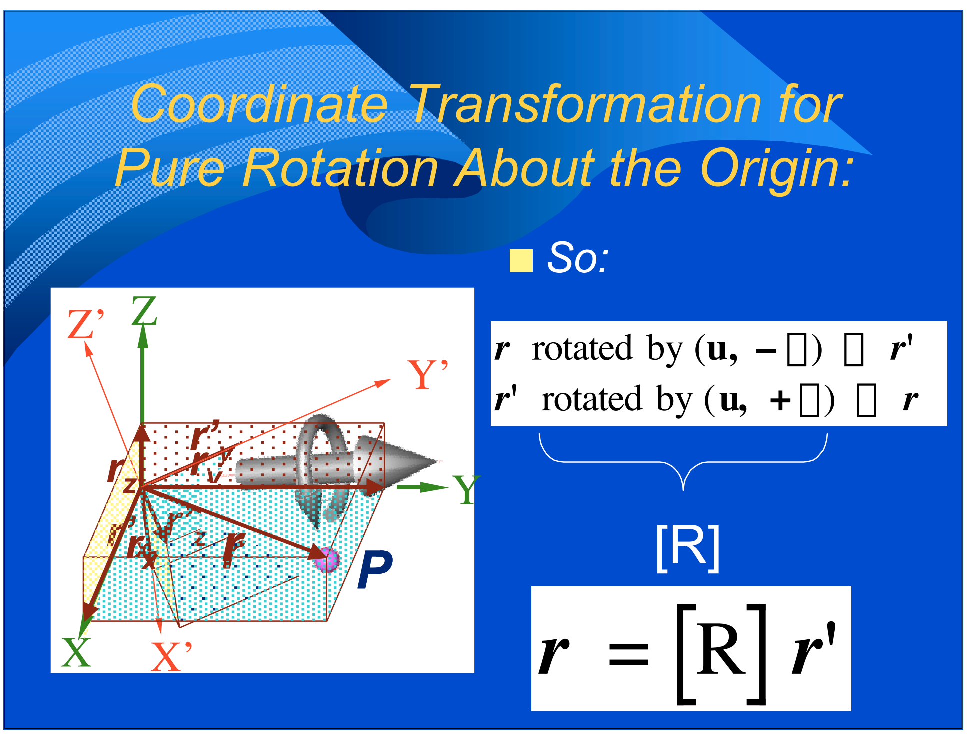
$$\boldsymbol{r}' = [\mathrm{T}]\,\boldsymbol{r}$$

Coordinate Transformation for Pure Rotation About the Origin:



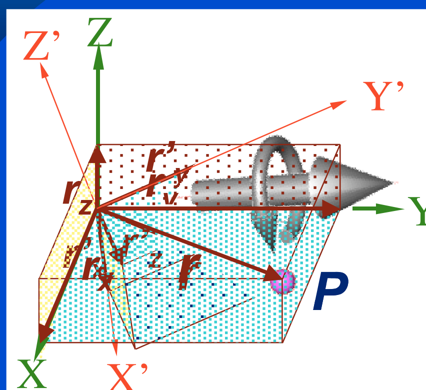
- Notice that rotating the coordinates by +□ is essentially the same as rotating the point P by -□


Coordinate Transformation for
Pure Rotation About the Origin:
Thus,
r rotated by (u, – □) □ r'
r' rotated by (u, + □) □ r
[R]
Z'
Z
Y'
Y
X
X'
P


Coordinate Transformation for
Pure Rotation About the Origin:
So:
r rotated by (u, – □) □ r'
r' rotated by (u, + □) □ r
[R]
r = [R] r'
Z'
Z
Y'
Y
X
X'
P

Coordinate Transformation for Pure Rotation About the Origin:

- *In other words:*



$$\boldsymbol{r} = [R]\,\boldsymbol{r}'$$

$$\boldsymbol{r}' = [R]^{-1}\,\boldsymbol{r}$$

Coordinate Transformation for Pure Rotation About the Origin:

$$[T] = [R]^{-1} = [R]^{T}$$

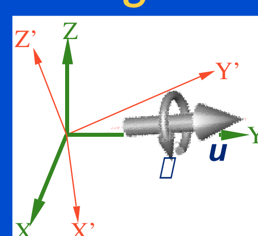
Coordinate Transformation for Pure Rotation About the Origin:

$$[T] = [R]^{-1} = [R]^{T}$$

$$[T][R] = [R][T] = [I]$$

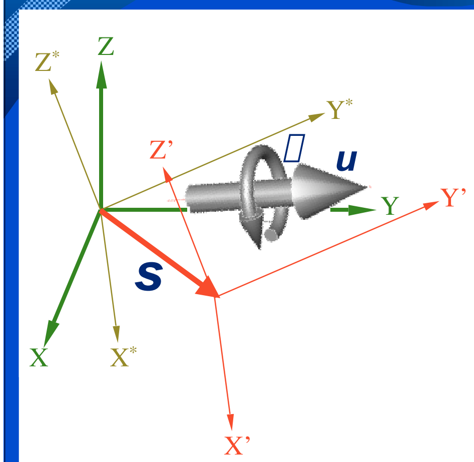
Coordinate Transformation for Pure Rotation About the Origin:

$$[T] =$$



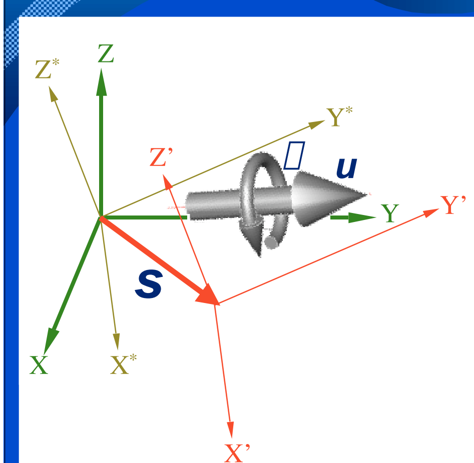
$$\begin{bmatrix} vers\,\theta\, u_x^2 + cos\,\theta & vers\,\theta\, u_x u_y + sin\,\theta\, u_z & vers\,\theta\, u_x u_z - sin\,\theta\, u_y \\ vers\,\theta\, u_x u_y - sin\,\theta\, u_z & vers\,\theta\, u_y^2 + cos\,\theta & vers\,\theta\, u_y u_z + sin\,\theta\, u_x \\ vers\,\theta\, u_x u_z + sin\,\theta\, u_y & vers\,\theta\, u_y u_z - sin\,\theta\, u_x & vers\,\theta\, u_z^2 + cos\,\theta \end{bmatrix}$$

Coordinate Transformation for Pure Rotation About the Origin Followed by a Translation



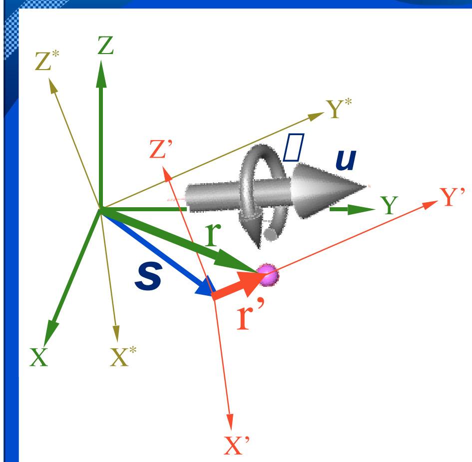
- *We are given the XYZ starting coordinate system shown in green.*
- *Also given is the rotation axis vector **u** (through the origin) and the angle θ*
- *We are also given a displacement vector **s***

Coordinate Transformation for Pure Rotation About the Origin Followed by a Translation



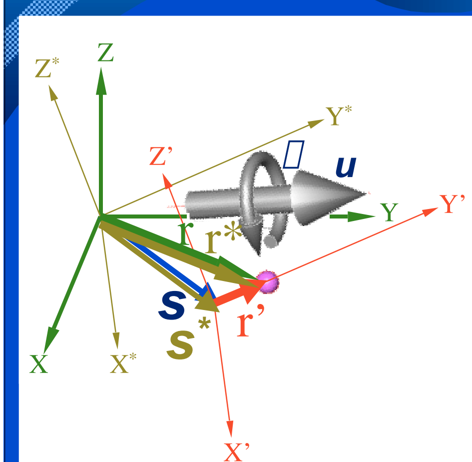
- *After the rotation the coordinate system becomes $X^* Y^* Z^*$*
- *After the subsequent translation it becomes X' Y' Z'*

Coordinate Transformation for Pure Rotation About the Origin Followed by a Translation



- *Given the point P in the original system located by **r** we want to find the components of **r'***

Coordinate Transformation for Pure Rotation About the Origin Followed by a Translation

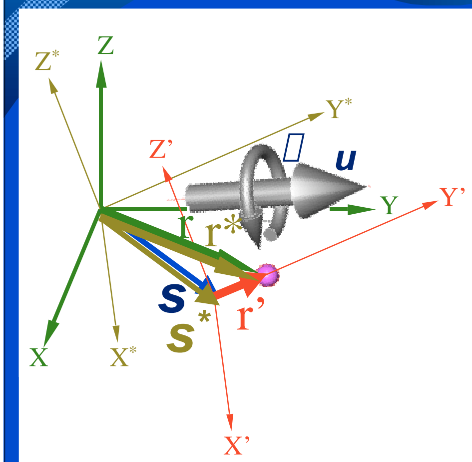


$$\boldsymbol{r}^* = \left[\mathrm{T}\right] r$$

$$\boldsymbol{s}^* = \left[\mathrm{T}\right] \boldsymbol{s}$$

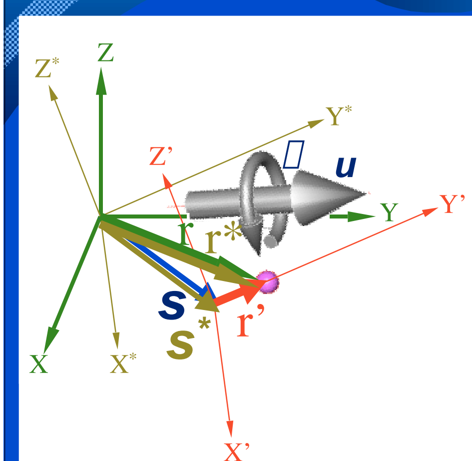
- *Here, [T] is the usual coordinate transformation matrix for a pure rotation about the origin.*

Coordinate Transformation for Pure Rotation About the Origin Followed by a Translation



- *Now translate the starred coordinate system by* $\boldsymbol{s}^*$
- *This is the same as translating P by* $-\boldsymbol{s}^*$

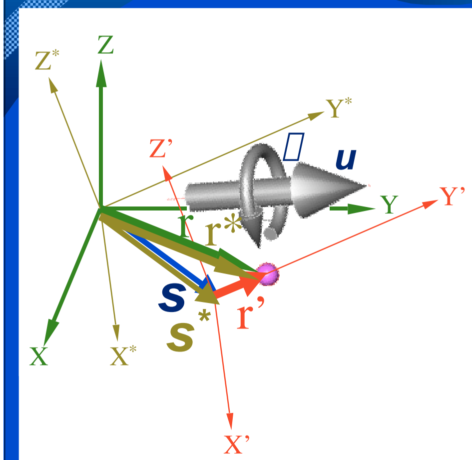
Coordinate Transformation for Pure Rotation About the Origin Followed by a Translation



- *So*

$$\boldsymbol{r}' = \boldsymbol{r}^* - \boldsymbol{s}^* = [\mathrm{T}]\,\boldsymbol{r} - [\mathrm{T}]\,\boldsymbol{s}$$

Coordinate Transformation for Pure Rotation About the Origin Followed by a Translation



- *In 4 X 4 matrix form (Gee, it sounds like an SUV) we have:*

$$\boldsymbol{r}' = [\mathrm{A}]\,\boldsymbol{r}$$

where [A] is a four by four matrix.

Coordinate Transformation for Pure Rotation About the Origin Followed by a Translation

- *So:*

$$\begin{bmatrix} 1 \\ r'_x \\ r'_y \\ r'_z \end{bmatrix} = \left[\begin{array}{c|c} \mathrm{I} & 0\;0\;0 \\ \hline -[\mathrm{T}]\,\boldsymbol{s} & [\mathrm{T}] \end{array}\right] \begin{bmatrix} 1 \\ r_x \\ r_y \\ r_z \end{bmatrix}$$

Coordinate Transformation for Pure Rotation About the Origin Followed by a Translation

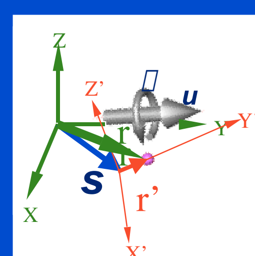
- *or:*

$$\boldsymbol{r}' = \left[\begin{array}{c:c} \mathrm{I} & 0 \\ \hdashline -[\mathrm{T}]\,\boldsymbol{s} & [\mathrm{T}] \end{array}\right] \boldsymbol{r}$$

Column Submatrix

Coordinate Transformation for Pure Rotation About the Origin Followed by a Translation

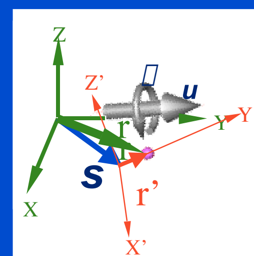
$$\boldsymbol{r}' = \underbrace{\left[\begin{array}{c:c} \mathrm{I} & 0 \\ \hdashline -[\mathrm{T}]\,\boldsymbol{s} & [\mathrm{T}] \end{array}\right]}_{[\mathrm{A}]} \boldsymbol{r}$$



- *[A] represents a rotation of the coordinate system by T about its origin followed by a translation by **s** specified in the <u>original</u> system!*

Coordinate Transformation for Pure Rotation About the Origin Followed by a Translation

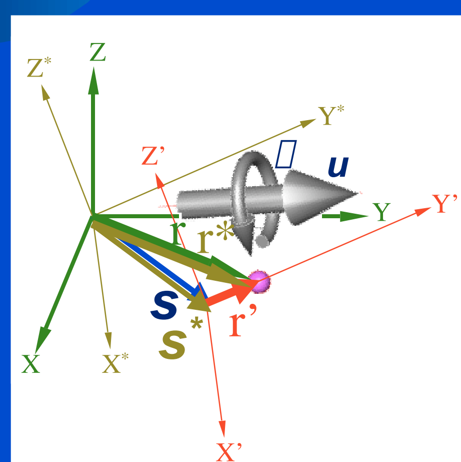
$$\mathbf{r}' = \underbrace{\left[\begin{array}{c:c} \mathrm{I} & 0 \\ \hdashline -[\mathrm{T}]\,\mathbf{s} & [\mathrm{T}] \end{array}\right]}_{[A]} \mathbf{r}$$



- *How can we invert this 4X4 transformation matrix [A] that we just derived?*

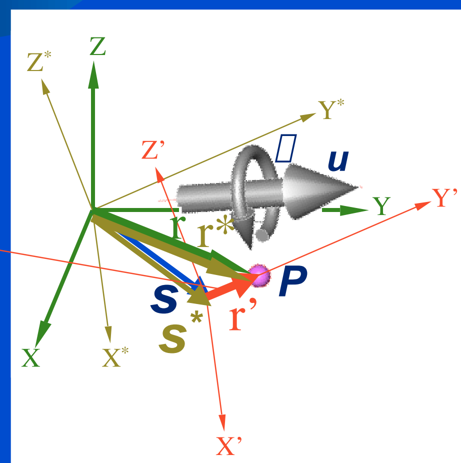
Inversion of 4X4 Matrices

- *[A] represents a rotation of the coordinate system by T about its origin followed by a translation by **s** specified in the original system.*



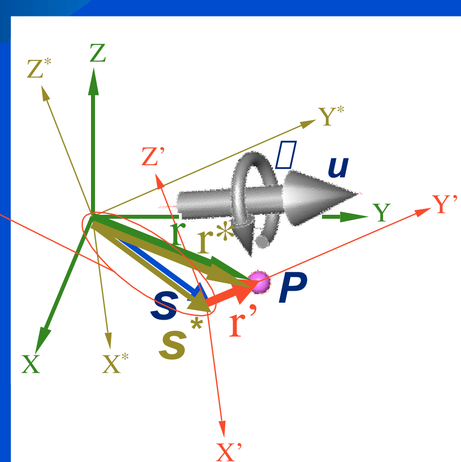
Inversion of 4X4 Matrices

- *A typical point P is defined in the current coordinate system x' y' z' by position vector **r'**.*



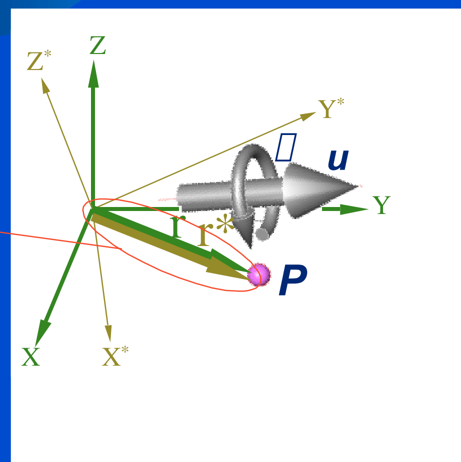
Inversion of 4X4 Matrices

- *To invert this matrix [A] physically we do the following:*
  - *Shift back by **s*** defined in the transformed system*



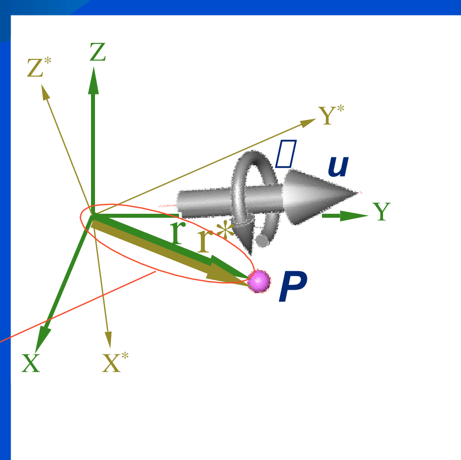
Inversion of 4X4 Matrices

- To invert this matrix [A] physically we do the following:
  - Shift back by $\mathbf{s}^*$ defined in the transformed system (this gives us $\mathbf{r}^*$ in the $x^* y^* z^*$ system.)



Inversion of 4X4 Matrices

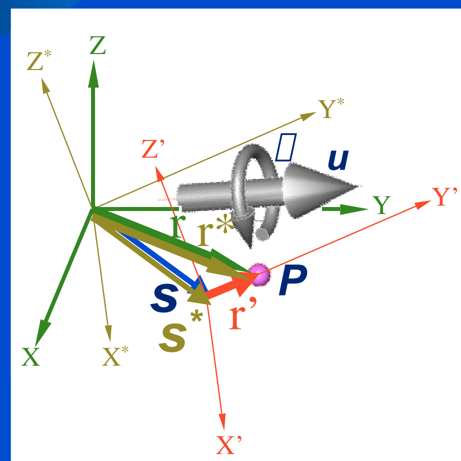
- To invert this matrix [A] physically we do the following:
  - Shift back by $\mathbf{s}^*$ defined in the transformed system (this gives us $\mathbf{r}^*$ in the $x^* y^* z^*$ system.)
  - Rotate back to obtain the original $\mathbf{r}$ in the x y z system.



Inversion of 4X4 Matrices

- *Mathematically, here's what happens:*
  - *Shift back by **s*** defined in the transformed system (this gives us **r*** in the x* y* z* system.)*
- *In other words,*

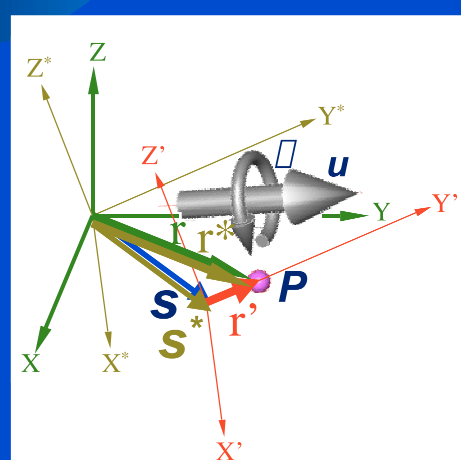
$$\boldsymbol{r}^* = [\mathrm{T}]\,\boldsymbol{s} + \boldsymbol{r}'$$



Inversion of 4X4 Matrices

- *And*

$$\begin{aligned}\boldsymbol{r} &= [\mathrm{T}]^{-1}\,\boldsymbol{r}^* \\ &= [\mathrm{T}]^{-1}\left([\mathrm{T}]\,\boldsymbol{s} + \boldsymbol{r}'\right) \\ &= \boldsymbol{s} + [\mathrm{T}]^{-1}\boldsymbol{r}'\end{aligned}$$

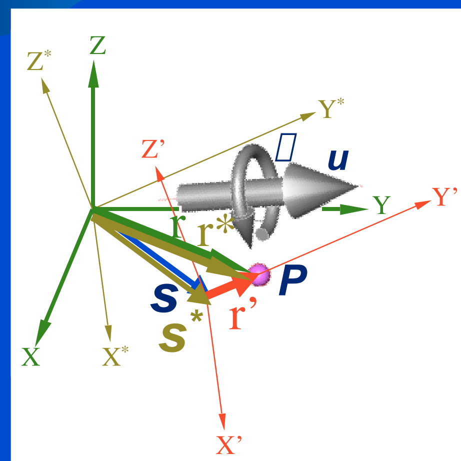


Inversion of 4X4 Matrices

- So

$$[T]^{-1} = [R]$$

$$[A]^{-1} = [M] = \left[\begin{array}{c:c} 1 & 0 \\ \hdashline \boldsymbol{s} & [R] \end{array}\right]$$



Inversion of 4X4 Matrices

- In general, to invert a 4X4 displacement matrix:

$$\underbrace{\left[\begin{array}{c:c} 1 & 0 \\ \hdashline \boldsymbol{s} & [R] \end{array}\right]^{-1}}_{[M]^{-1}} = \left[\begin{array}{c:c} 1 & 0 \\ \hdashline -[R]^{-1}\boldsymbol{s} & [R]^{-1} \end{array}\right] = \underbrace{\left[\begin{array}{c:c} 1 & 0 \\ \hdashline -[T]\boldsymbol{s} & [T] \end{array}\right]}_{[A]}$$

Inversion of 4X4 Matrices

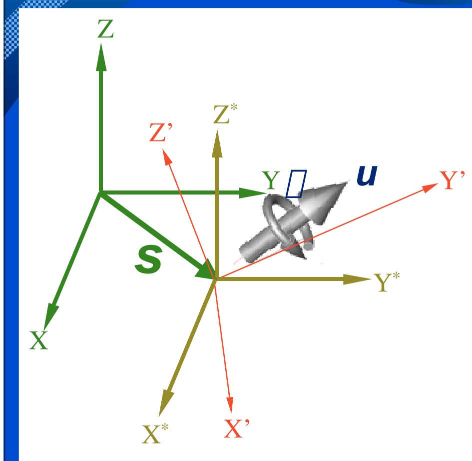
- *For the special case of a screw motion:*

$$[\mathrm{T}]\,\boldsymbol{s} = [\mathrm{R}]\,\boldsymbol{s} = \boldsymbol{s}$$

Inversion of 4X4 Matrices

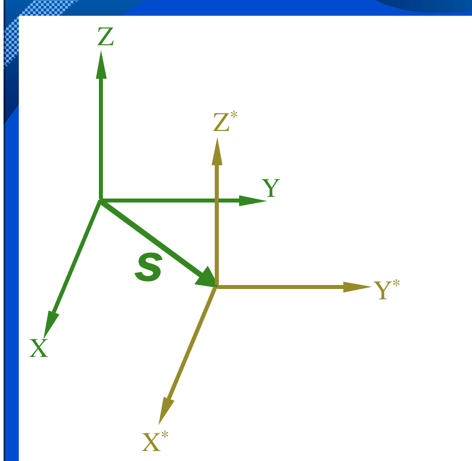
- *Note: The product of 4X4 screw matrices is not necessarily a screw matrix!*

Coordinate Transformation for Translation Followed by a Rotation About the New Origin



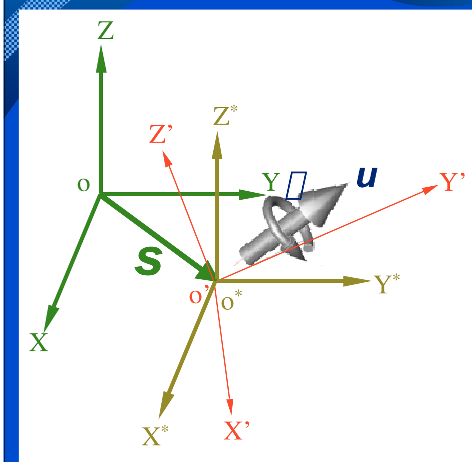
- *We are given the XYZ starting coordinate system shown in green.*
- *Displacement vector **s** is also given, defined in the original coordinate system.*
- *In addition we have the given rotation **u**, □ which will take place about the displaced origin.*

Coordinate Transformation for Translation Followed by a Rotation About the New Origin



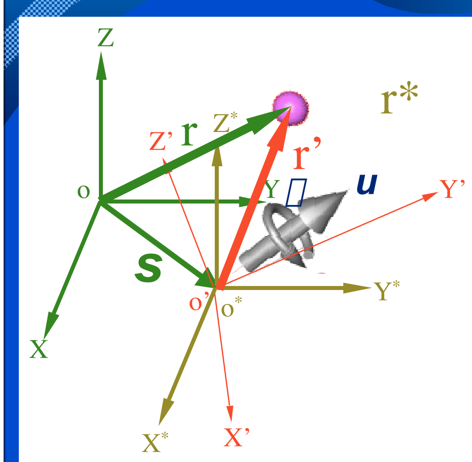
- *After translation by vector **s** we are brought to the coordinate system x* y* z* as shown:*

Coordinate Transformation for Translation Followed by a Rotation About the New Origin



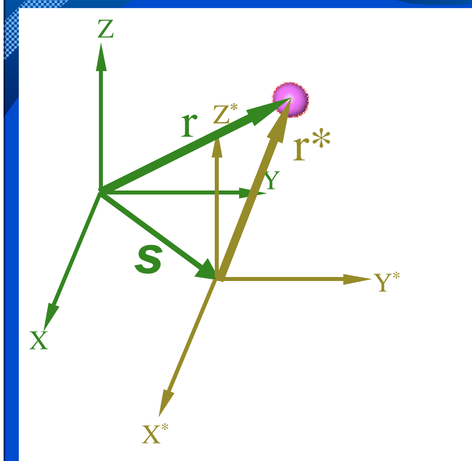
- *We then rotate the starred coordinate system to the x' y' z' final system by rotating about the new origin o* (which = o', the final origin).*
- *The rotation takes place about **u** (through the new origin) and by the angle θ.*

Coordinate Transformation for Translation Followed by a Rotation About the New Origin



- *We want to find the components of **r'** in the translated and rotated coordinate system, knowing the original **r***

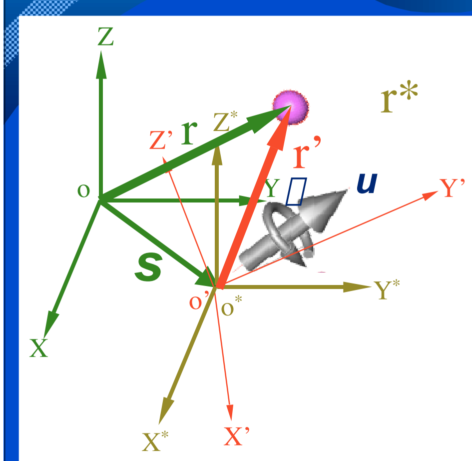
Coordinate Transformation for Translation Followed by a Rotation About the New Origin



- *Step 1: Translate xyz by vector* ***s*** *to x* y* z*.*
- *(This is accomplished by translating P by -**s**)*

$$\boldsymbol{r}^{*} = \boldsymbol{r} - \boldsymbol{s}$$

Coordinate Transformation for Translation Followed by a Rotation About the New Origin



- *Step 2: Rotate $x^* y^* z^*$ by [T] matrix:*

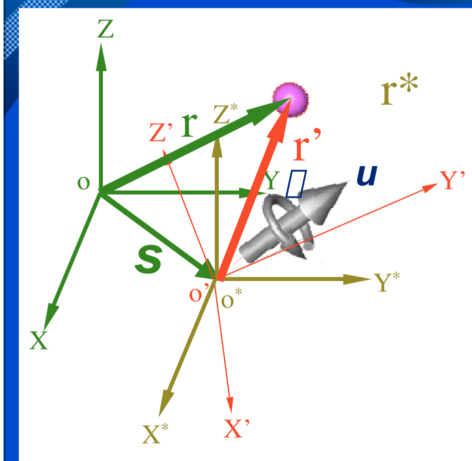
$$\boldsymbol{r}' = [\mathrm{T}]\,\boldsymbol{r}^{*}$$

$$\boldsymbol{r}' = [\mathrm{T}]\,(\boldsymbol{r} - \boldsymbol{s})$$

$$\boldsymbol{r}' = [\mathrm{T}]\,\boldsymbol{r} - [\mathrm{T}]\,\boldsymbol{s}$$

- *This is as before, since the rotation is about the new origin.*

Coordinate Transformation for Translation Followed by a Rotation About the New Origin



- In other words,

$$\boldsymbol{r}' = [\mathrm{T}]\,\boldsymbol{r} - [\mathrm{T}]\,\boldsymbol{s}$$

$$\boldsymbol{r}' = [\mathrm{A}]\,\boldsymbol{r}$$

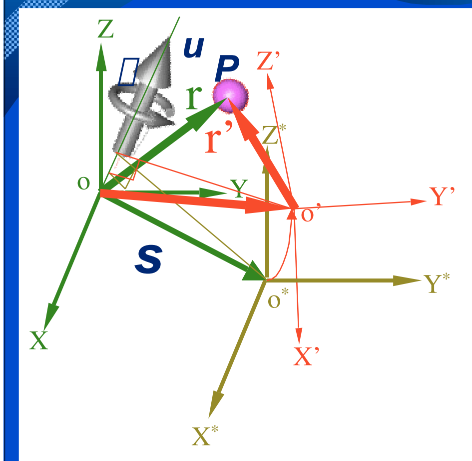
$$[\mathrm{A}] = \left[\begin{array}{c|c} 1 & 0 \\ \hline -[\mathrm{T}]\,\boldsymbol{s} & [\mathrm{T}] \end{array}\right]$$

Coordinate Transformation for Translation Followed by a Rotation About the New Origin

$$[\mathrm{A}] = \left[\begin{array}{c|c} 1 & 0 \\ \hline -[\mathrm{T}]\,\boldsymbol{s} & [\mathrm{T}] \end{array}\right]$$

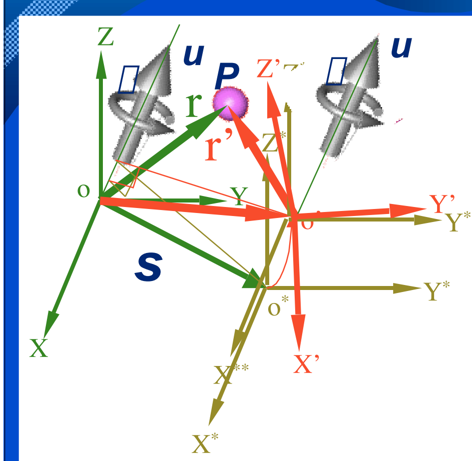
- *Thus, this matrix describes either*
  - *a coordinate transformation by rotation followed by translation defined in the original system*
  - *or a coordinate transformation produced by translating by* ***s*** *defined in the original system and followed by a rotation about the new origin (in either the * or the ' system).*

Coordinate Transformation for Translation Followed by a Rotation About the Original Origin



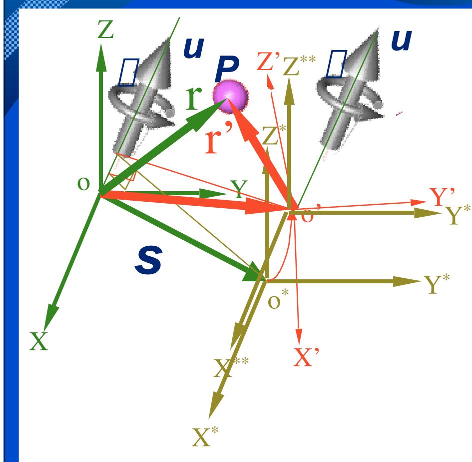
- By inspection, we see that this motion is equivalent to a translation from o to o' followed by a rotation about **u** through the new origin o' by angle □

Coordinate Transformation for Translation Followed by a Rotation About the Original Origin



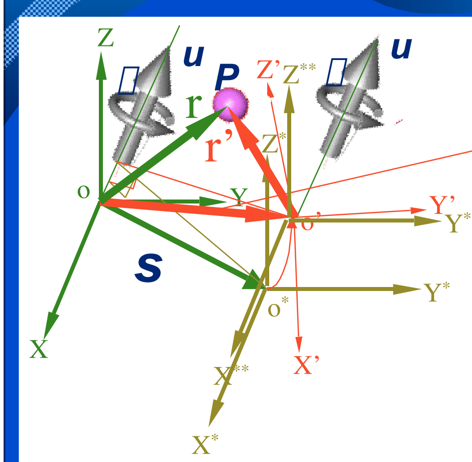
- By inspection, we see that this motion is equivalent to a translation from o to o' followed by a rotation about **u** through the new origin o' by angle □

Coordinate Transformation for Translation Followed by a Rotation About the Original Origin



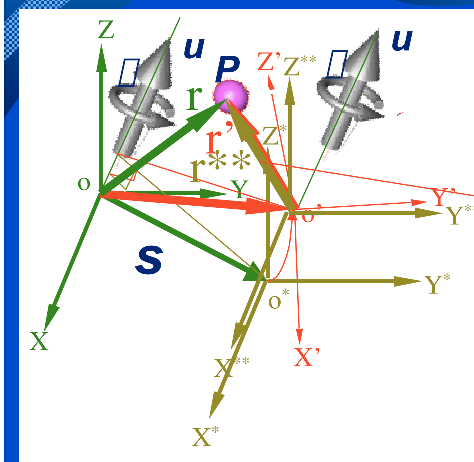
- *By inspection, we see that this motion is equivalent to a translation from o to o' followed by a rotation about **u** through the new origin o' by angle θ*

Coordinate Transformation for Translation Followed by a Rotation About the Original Origin



- *Translation from o to o' is accomplished by [R]**s** or $[T]^{-1}$**s***
- *So translation of the coordinates by $[T]^{-1}$**s** has the same effect as translating P by $-[T]^{-1}$**s***

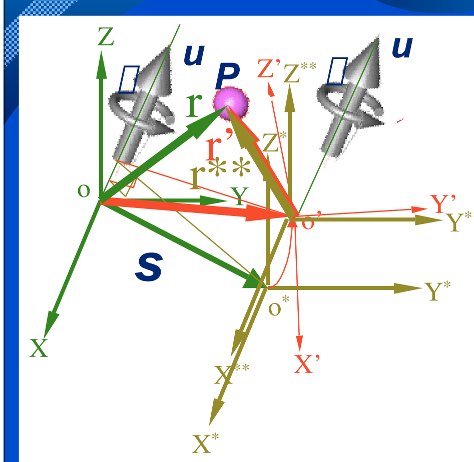
Coordinate Transformation for Translation *Followed* by a Rotation About the *Original* Origin



- So in the ** coordinate system point P is given by

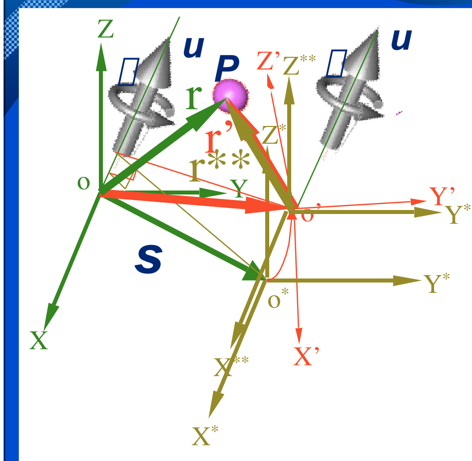
$$\boldsymbol{r}^{**} = -[T]^{-1}\boldsymbol{s} + \boldsymbol{r}$$

Coordinate Transformation for Translation *Followed* by a Rotation About the *Original* Origin



- Rotating the coordinate system about the new origin o' is accomplished by operating on the $\boldsymbol{r}^{**}$ by [T]

Coordinate Transformation for Translation Followed by a Rotation About the Original Origin



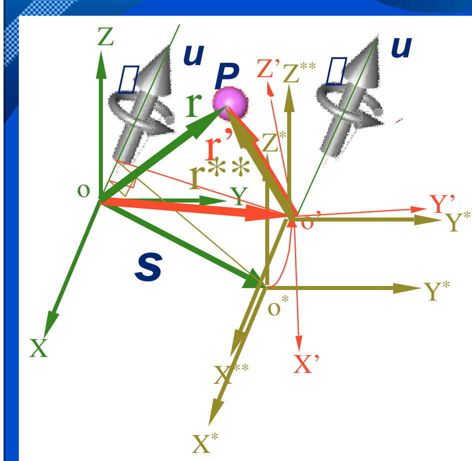
- *That means that*

$$\boldsymbol{r}' = [T]\left(\boldsymbol{r} - [T]^{-1}\boldsymbol{s}\right)$$

$$= [T]\boldsymbol{r} - \boldsymbol{s}$$

$$\begin{bmatrix} r'_x \\ r'_y \\ r'_z \end{bmatrix} = [T]\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} - \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix}$$

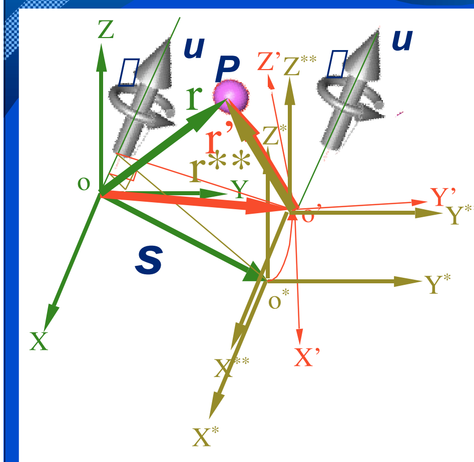
Coordinate Transformation for Translation Followed by a Rotation About the Original Origin



- *In 4X4 form that gives*

$$\begin{bmatrix} 1 \\ r'_x \\ r'_y \\ r'_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -s_x & & & \\ -s_y & & [T] & \\ -s_z & & & \end{bmatrix} \begin{bmatrix} 1 \\ r_x \\ r_y \\ r_z \end{bmatrix}$$

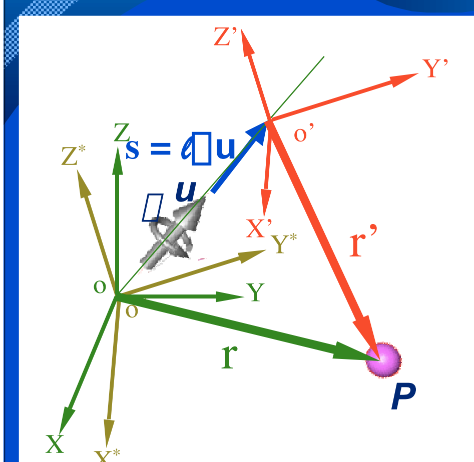
Coordinate Transformation for Translation Followed by a Rotation About the Original Origin



- or

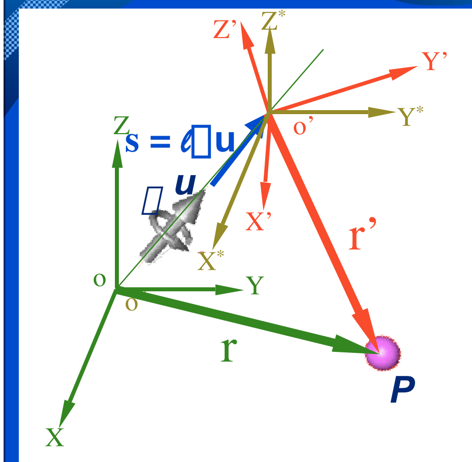
$$\mathbf{r}' = \underbrace{\left[\begin{array}{c|c} 1 & 0 \\ \hline -\mathbf{s} & T \end{array}\right]}_{[A]} \mathbf{r}$$

Coordinate Transformation by a Screw Motion



- Here, both **u** and **s** are defined in the xyz coordinate system.

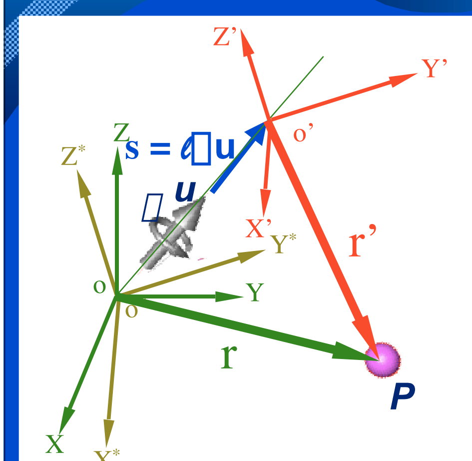
Coordinate Transformation by a Screw Motion



- *If we consider this motion as a translation followed by a rotation we get*

$$\boldsymbol{r}' = \underbrace{\begin{bmatrix} 1 & 0 \\ -\boldsymbol{s} & T \end{bmatrix}}_{[A]} \boldsymbol{r}$$

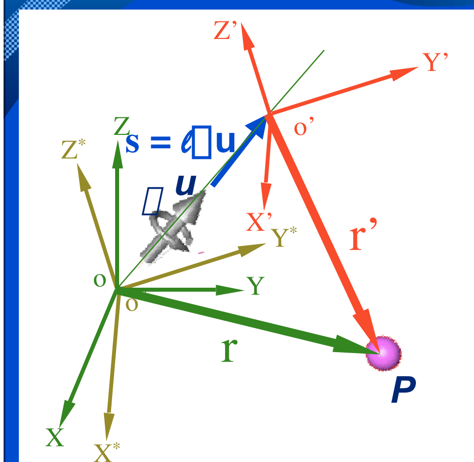
Coordinate Transformation by a Screw Motion



- *If we think of it as a rotation followed by a translation we get*

$$\begin{aligned} \boldsymbol{r}^* &= [T]\,\boldsymbol{r} \\ \boldsymbol{r}' &= \boldsymbol{r}^* - \boldsymbol{s}^* \\ &= [T]\,\boldsymbol{r} - [T]\,\boldsymbol{s} \\ &= [T]\,(\boldsymbol{r} - \boldsymbol{s}) \end{aligned}$$

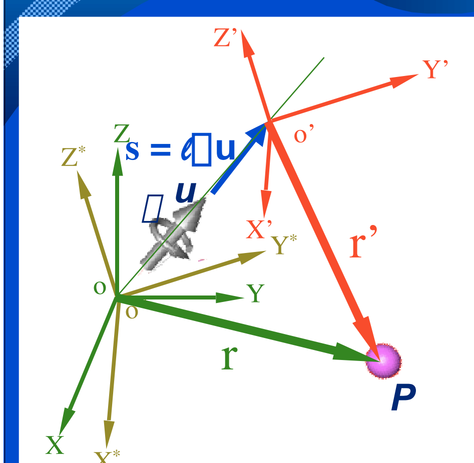
Coordinate Transformation by a Screw Motion



- *Putting this in 4X4 matrix form gives:*

$$\boldsymbol{r}' = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline -\boldsymbol{Ts} & \boldsymbol{T} \end{array}\right]\boldsymbol{r}$$

Coordinate Transformation by a Screw Motion

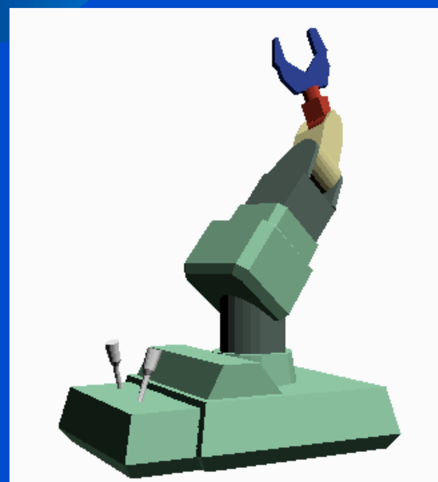


- *But remember that **s** is along the axis of rotation and is invariant under this rotation!*
- *This means we can again write this as:*

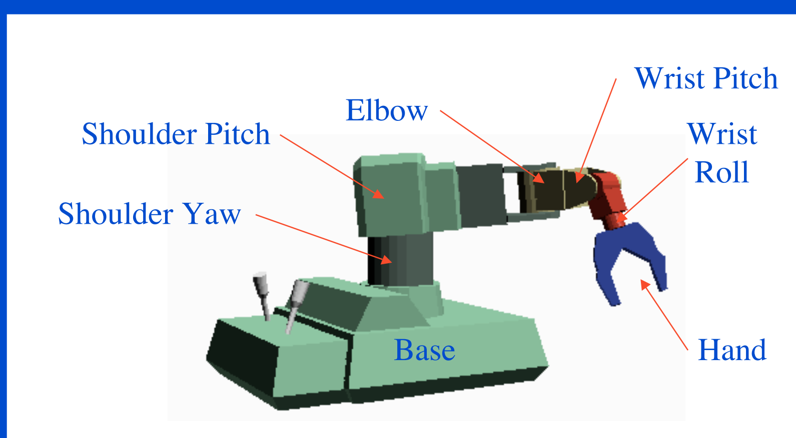
$$\boldsymbol{r}' = \underbrace{\left[\begin{array}{c|c} 1 & 0 \\ \hline -\boldsymbol{s} & T \end{array}\right]}_{[A]}\boldsymbol{r}$$

Putting it all together: Armatron Robot Example

- Suppose we wish to develop a kinematic computer model of the Armatron toy robot shown here.
- This is an example of an "Open Loop" spatial mechanism.
- It has how many degrees of freedom?

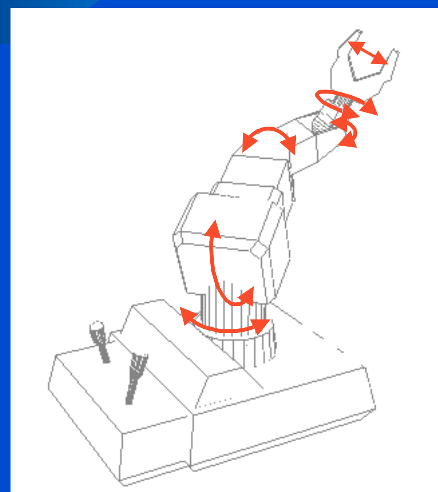


Putting it all together: Armatron Robot Example



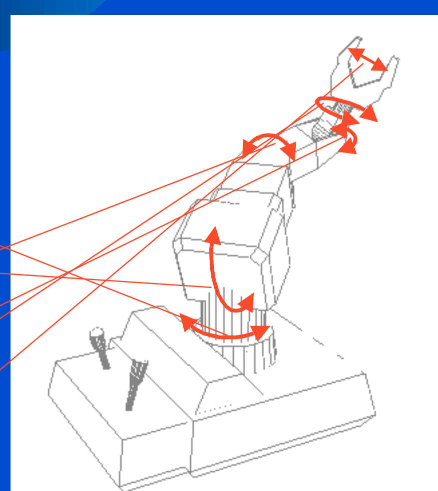
Putting it all together: Armatron Robot Example

- It appears to have six degrees of freedom as shown:



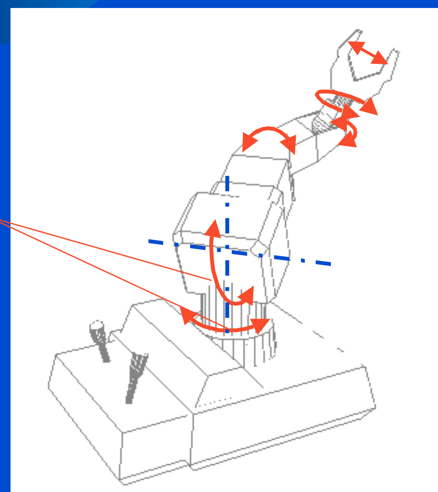
Putting it all together: Armatron Robot Example

- It appears to have six degrees of freedom as shown:
- It has a
  - Shoulder yaw joint
  - Shoulder pitch joint
  - Elbow joint
  - Wrist pitch joint
  - Wrist roll joint
  - Hand grasp joint



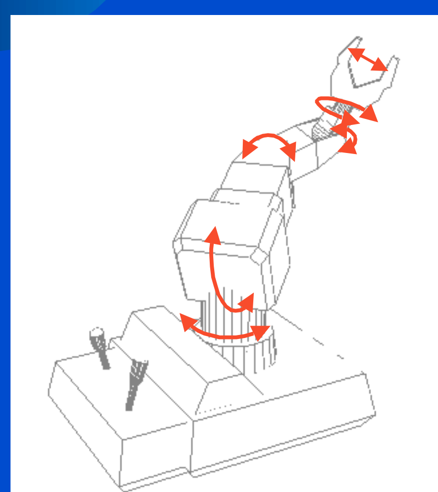
Putting it all together: Armatron Robot Example

- With the exception of the hand joint all of these are turning joints.
- Several of the joints (such as the two shoulder joints) have intersecting axes so as to simplify the structure.



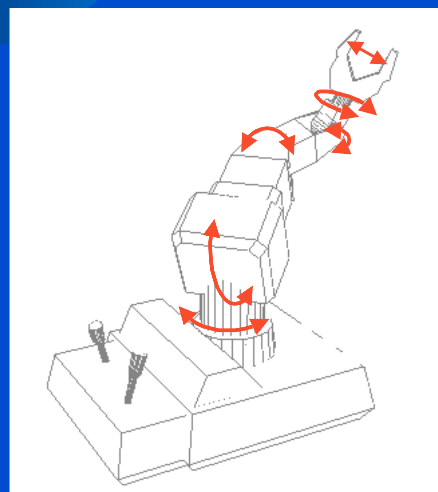
Putting it all together: Armatron Robot Example

- How can we model this on the computer?



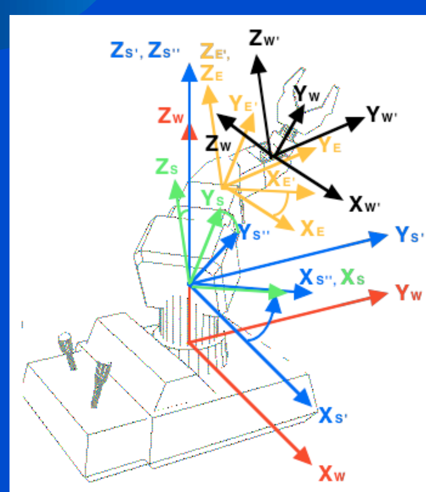
Putting it all together: Armatron Robot Example

- First, let's establish a convenient system of coordinate systems.
- We'll attach at least one coordinate system to each part.



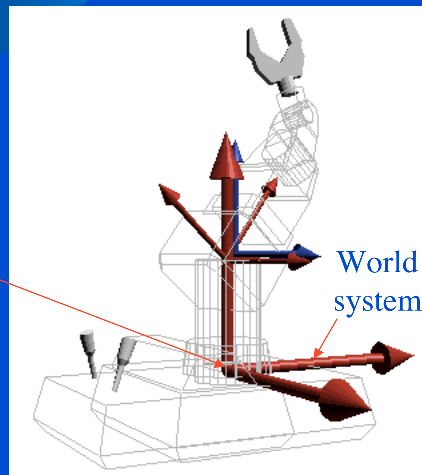
Putting it all together: Armatron Robot Example

- In fact, let's sprinkle in coordinate systems whenever it seems it might make life simpler!



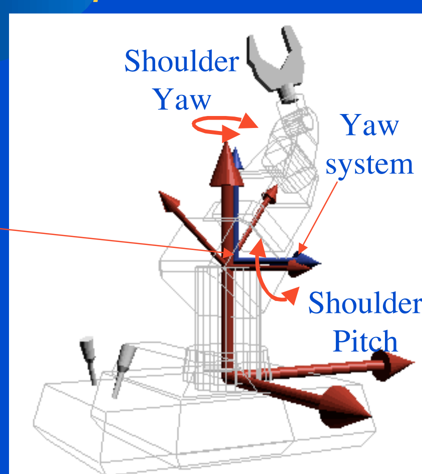
Putting it all together: Armatron Robot Example

- For instance, we probably want to be able to describe where objects are in terms of a convenient world coordinate system fixed in the robot's base.



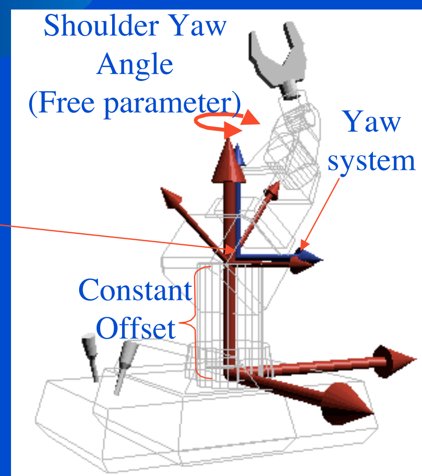
Putting it all together: Armatron Robot Example

- But it would be a lot easier to describe the pitch of the shoulder joint in terms of a coordinate system like the blue system shown here.
- The blue system's z axis is always vertical, aligned with the shoulder's yaw joint, but the x axis always points along the shoulder's pitch joint.



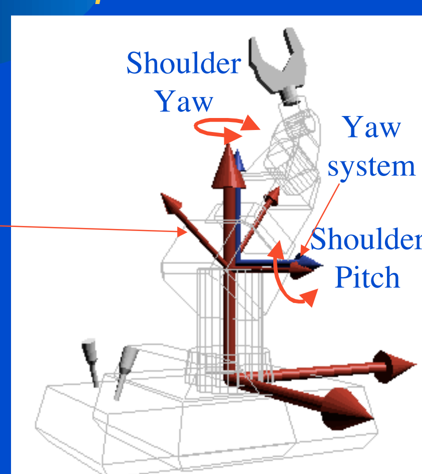
Putting it all together: Armatron Robot Example

- *The blue system's position relative to the world system is completely specified by the shoulder yaw angle variable and the offset along the z axis which is a constant.*



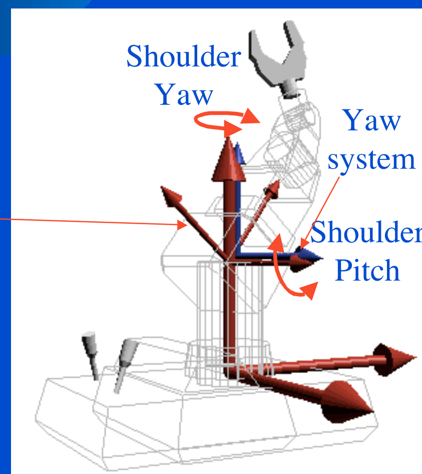
Putting it all together: Armatron Robot Example

- *Similarly, it is easy to describe the geometry of the shoulder object itself in terms of a coordinate system that is fixed in the object, such as this red system.*
- *It is easy to describe where that system is with respect to the blue yaw system if we line up their x axes and their origins.*



Putting it all together: Armatron Robot Example

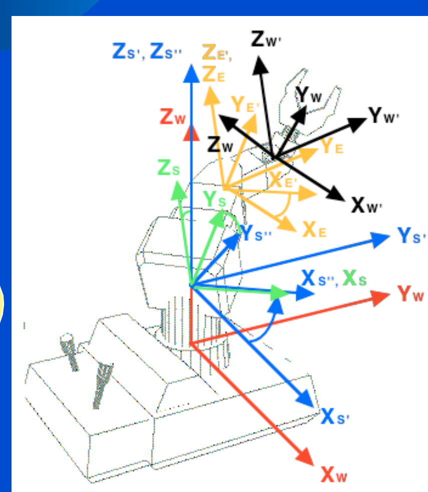
- *Then the shoulder pitch angle is all we need to know in order to figure out where the shoulder is with respect to the blue system.*



Putting it all together: Armatron Robot Example

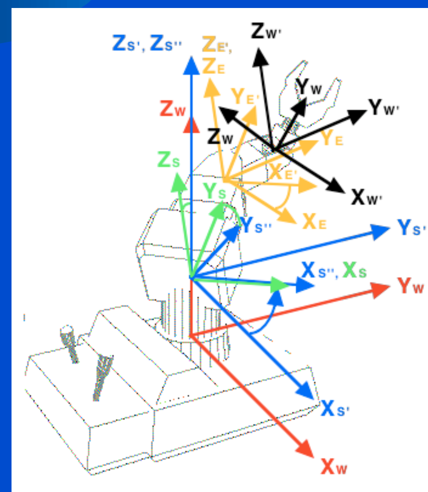
- *Let's attach two coordinate systems to each turning joint!*
- *We'll f*

Good grief!
That's a lot of
coordinate systems!



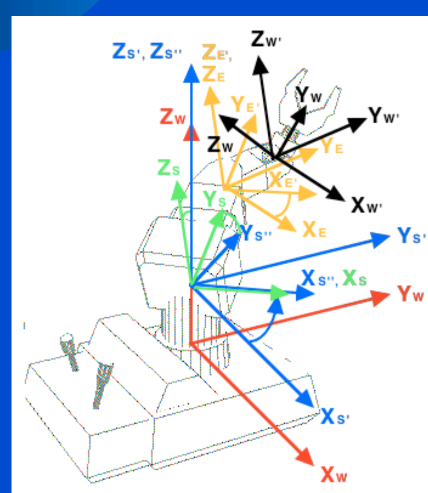
Putting it all together: Armatron Robot Example

- It is a lot of coordinate systems but it will actually make our life easier!
- We will cleverly pick our coordinate systems so they line up with joint axes.



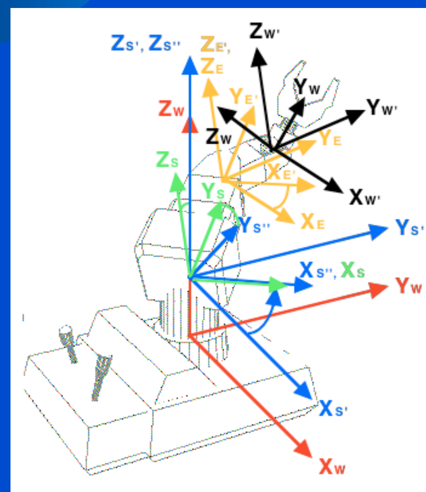
Putting it all together: Armatron Robot Example

- Within a single rigid link, a constant 4X4 transformation matrix can bring us from one coordinate system on the link to the next.



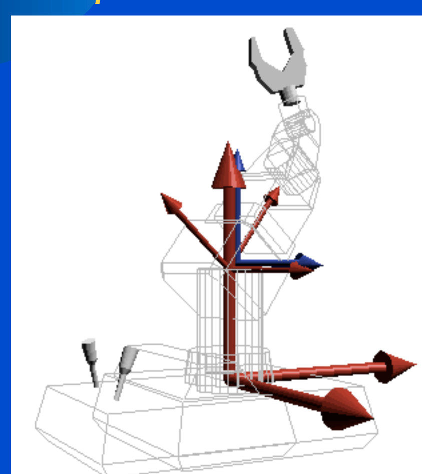
Putting it all together: Armatron Robot Example

- As you go across a joint, an easily written 4X4 transformation matrix (which is a function of the joint variable) can bring us from the coordinate system on one side of the joint to the one on the other side.



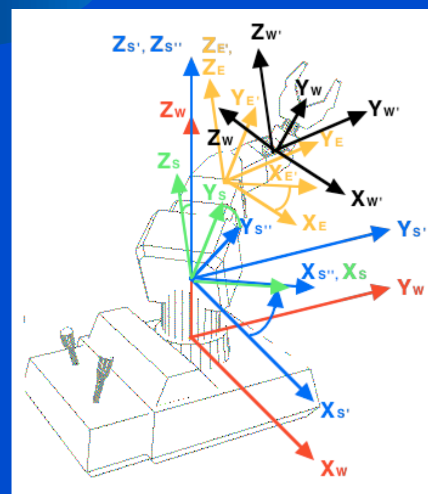
Putting it all together: Armatron Robot Example

- You'll want to have a convenient "World" coordinate system in which to describe the locations of parts.
- You'll also want to know where parts are in relation to the hand.



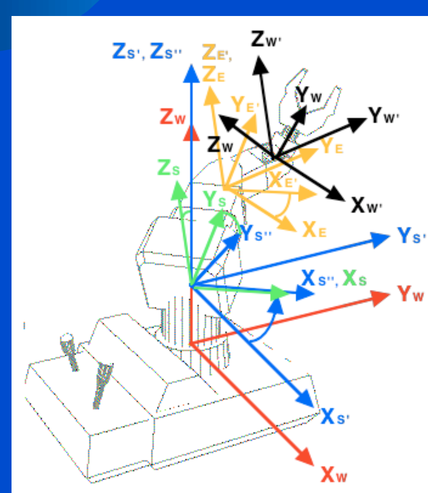
Putting it all together: Armatron Robot Example

- *You'll want to be able to work back and forth between the "World" coordinate system and the hand coordinate system.*



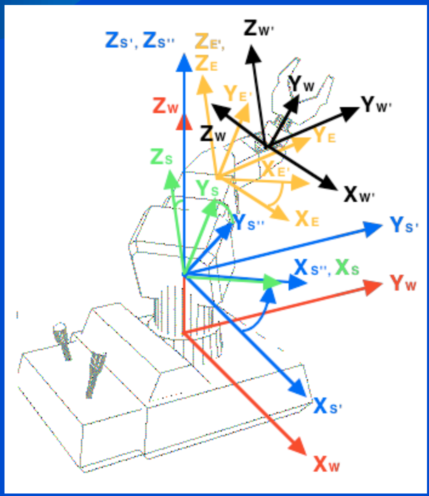
Putting it all together: Armatron Robot Example

- *This is just a matter of multiplying together strings of 4X4 matrices.*
- *Concatenate away.*
- *Just be sure you know whether to premultiply or to postmultiply!*



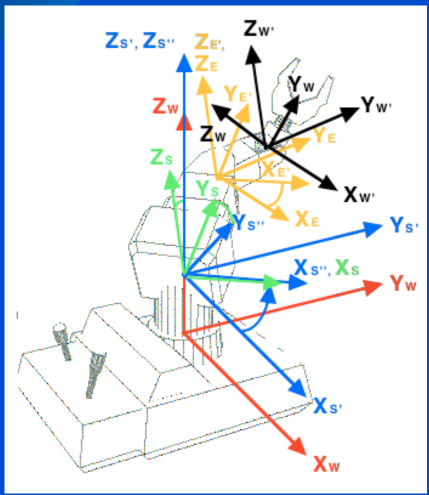
Putting it all together: Armatron Robot Example

- You'll want to know how things move around when you twiddle the various free parameters!
- You'll want to know where the various parts of the various links are to make sure that they don't collide!

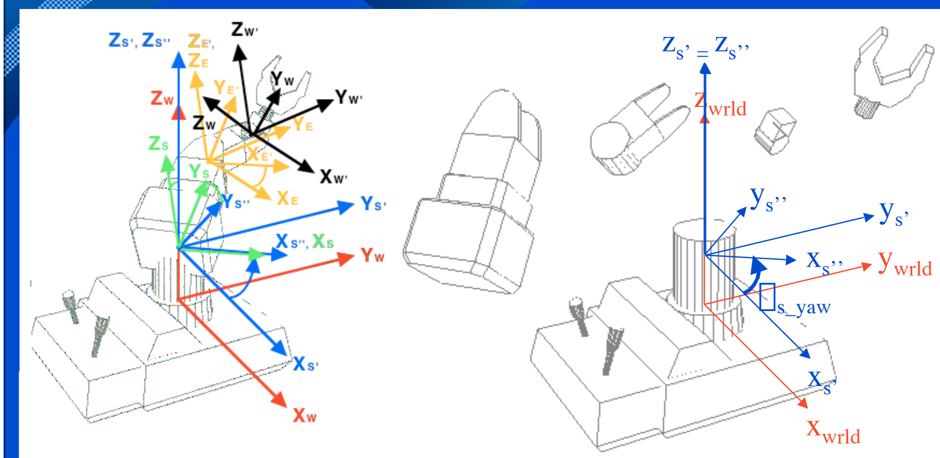


Putting it all together: Armatron Robot Example

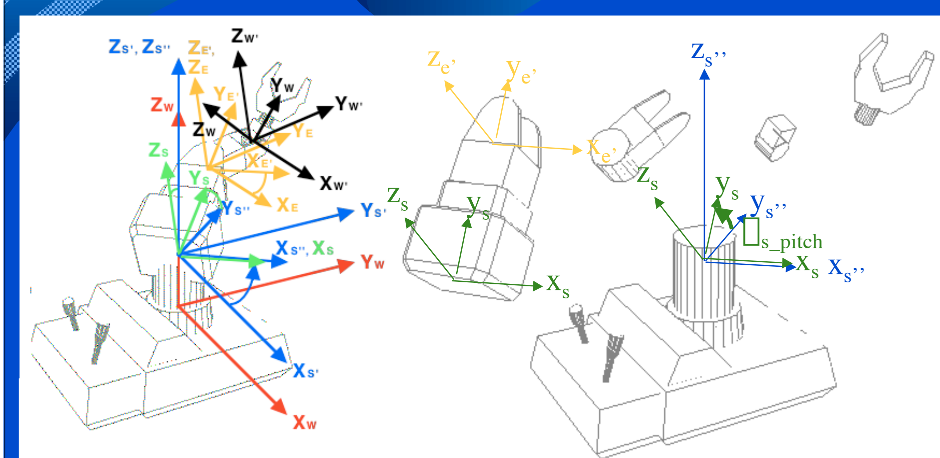
- We want to be sure we distinguish between situations in which
  - we merely express the same position of an object in a <u>different</u> coordinate system
  - and in which the object has moved within the <u>same</u> coordinate system!



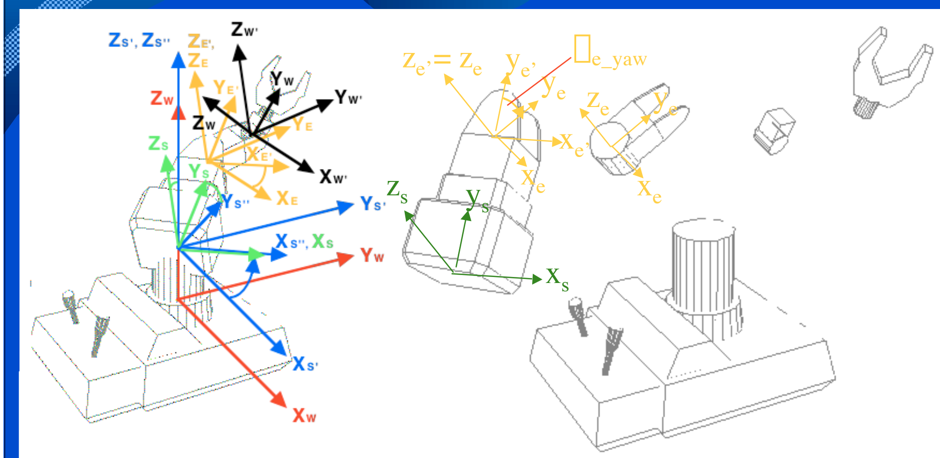
Putting it all together: Armatron Robot Example



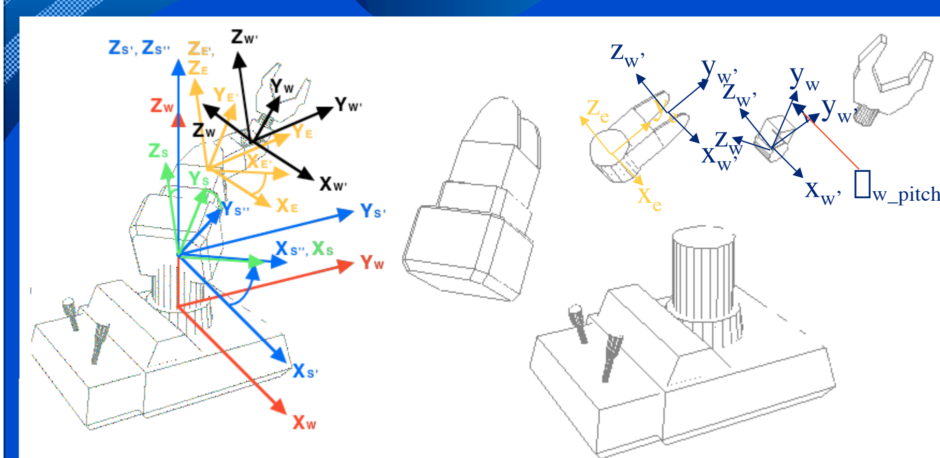
Putting it all together: Armatron Robot Example



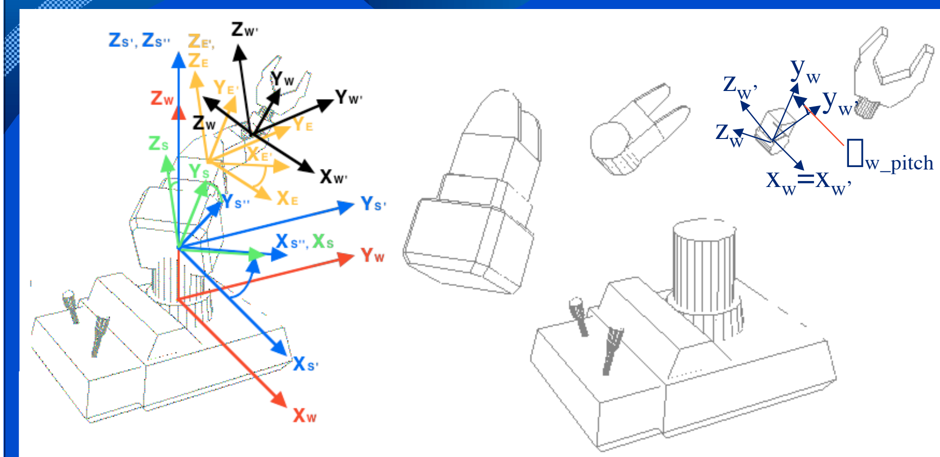
Putting it all together: Armatron Robot Example



Putting it all together: Armatron Robot Example



Putting it all together: Armatron Robot Example



Putting it all together: Armatron Robot Example